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Plastic design of pinned-base saw-tooth frames

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PLASTIC DESIGN OF PINNED-BASE SAW-TOOTH FRAMES

by

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A C K N O W L E D G E M E N T S

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William J. Eney is Head of the Department of Civil Engineering and Director of Fritz Engineering Laboratory.

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I. BASIC CONCEPTS

1.1 Introduction

As plastic design became more popular it was necessary to develop techniques to aid the designer and spare him from lengthy computations.

The upper and lower bound theorems are two approaches which allow the designer to find the ultimate load but it was still desirable to develop a method of determining the collapse load of a structure without investigating possible failure configurations which did not govern the design. In 1956, Ketter in his dissertation, presented a method by which design charts could be used to find the ultimate load, the collapse mode and member size for a particular shape frame and loading.(2)

The American Institute of Steel Construction published a handbook on plastic analysis and design in 1958 which contained design charts for certain types of pinned-base frames.(7)

In 1959, the Welding Research Council published a bulletin written by Ketter which presented design charts for the gable type frame.(5) Later that same year they published another bulletin by Ketter and Yen on the lean-to frame.(6)

This thesis is presented so that charts for the saw-tooth frame can be constructed, thereby making charts available for the most common type of mill buildings.

1.2 General Concepts and Assumption

Plastic analysis and design when properly used, will permit the designer to make more efficient use of materials which will result in a more economic design. Just as all theories and methods have their limitations, the designer must also recognize the limits to which a design based on "simple plastic theory" can be applied. In the following, a list of the assumptions made in this paper will give an indication of areas in which caution should be used in designing by this theory.

The method of Plastic Design is based upon determining the lowest possible load required to reduce a structure to a mechanism, through the development of zones of yielding at points of high moment. In the "simple plastic theory" the following assumptions are made:

1. Curvature increases indefinitely as the moment approaches the full plastic value.
2. Equilibrium can be formulated in the undeformed state.
3. Instability of the structure will not occur prior to the attainment of the ultimate load.
4. Shear and normal force influences are neglected.
5. Full continuity is provided by the connections so that the full plastic moment of the sections can be transmitted.
6. All loads increase in fixed proportions to one another.
7. Failure corresponds to the condition where the structure is reduced to a mechanism due to the formation of fully yielded zones or plastic hinges.

1.3 Upper and Lower Bound Theorems

Reference 1 shows that the following necessary and sufficient conditions must be met in order to have a valid solution based on "simple plastic theory."

1. The formation of a mechanism.

2. Equilibrium cannot be violated.

3. The moment may not exceed the absolute value of the plastic moment.

There has been developed two theorems which divide methods of plastic analysis into two groups. Those methods of solution which give an ultimate load which is greater than or at best equal to the true maximum carrying capacity, are considered upper bound methods and those which give solutions, which are lower than or at best equal to the ultimate load, are known as lower bound methods.

In the upper bound methods, equilibrium is satisfied and a mechanism is assumed thereby requiring a moment or plasticity check. The magnitudes of the maximum moments are adjusted until a mechanism is formed in the lower bound method of solution. Only a few of the more popular methods will be mentioned, here.

(a) Statical Method. The analysis of continuous beams and simple frames where the designer can visualize the probable failure mechanism prior to the analysis, are easily handled by this method; but as the number of degrees of redundancy increases or the structure becomes more complex, this method of analysis becomes very cumbersome. Since this is a lower bound method, it will be necessary to adjust the points of maximum moments until a mechanism is formed. This can be done very easily in simple frames and continuous beams by superimposing the moment diagram of the determinate structure upon the moment diagram of the structure loaded by the redundants and adjusting the magnitude of the maximum moments to allow the formation of a mechanism.

(b) Mechanism Method. Upper bounds to the correct load will be found by investigating all the possible collapse configurations which could occur for the loading on the structure. Since the structure will fail at its first opportunity, the configuration giving the lowest failure load will be the solution to the problem. In this method, a mechanism is assumed, equilibrium is formulated and the principal of virtual displacements is used to compute the ultimate load of the frame.

When the concept of instantaneous centers is used in conjunction with the mechanism method, it becomes a very powerful tool for analyzing the more complex type of structures. The mechanism method will be used in this paper to compute the collapse loads for both single and multiple span frames.

(c) Moment Balancing. A relaxation method of distributing moments, can be used in the analysis of certain types of frames. This method is particularly useful in making a plasticity check when a structure is found to be statically indeterminate after the formation of a mechanism. A considerable amount of trial and error is present when the moment-balancing method is used and experience is necessary to make efficient use of this method.

1.4 Ketter's Method

Reference 2 presents a method by which all the possible failure configurations of a particular type frame are investigated by the mechanism method, and the results are plotted in the form of non-dimensional design charts. For the multiple span structure a method called "Solution by Separation" was introduced thus enabling the development of design charts which were applicable to more complex types of frames. These charts take into account both dead and wind loading, and cover a range which includes all practical size structures and loadings.

Once the designer has decided upon the geometry and loading of the frame, the design charts will supply him with the information necessary to select a member size, the mode of collapse, and the location of the hinges which will form in the structure. These charts also enable the designer to proportion the structure such that the "least weight" solution can be found with comparatively little effort.

1.5 Purpose of This Study

The saw-tooth frame, or sometimes called the north-light portal is a frame usually used in mill buildings to take advantage of the additional window area made available by its geometry.

Design charts are already available for the pinned-base portal, gable and lean-to frames. This study was conducted for the purpose of deriving the necessary expressions for developing design charts which could be used for the pinned-base saw-tooth frame. The expressions presented are those for the multiple span case but can easily be modified to cover the single span case also.

The sample design charts presented in this thesis are constructed to enable the designer to use them in combination with those already available for other type pinned-base frames. By using the several charts together, the structure shown in Figure 1 can be easily analyzed.

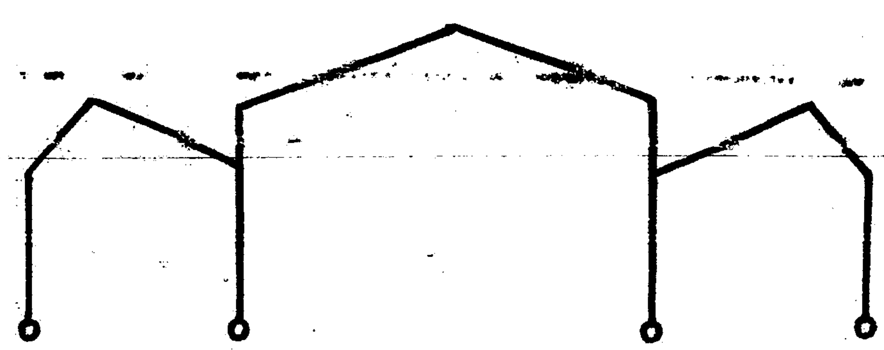

Figure 1

In chapter 4, a few frames will be analyzed to illustrate the method used in arriving at the solutions for the several different types of frames when combined into a multiple span structure.

II. PINNED-BASE SINGLE-SPAN SAW-TOOTH FRAMES

2.1 General Description

It was previously mentioned that the development of the design charts for the saw-tooth frame, would then make available, charts which would enable the designer to determine the ultimate load of a frame according to simple plastic theory for all of the commonly used shapes of mill buildings and frames.

The frames considered in this paper are assumed to be made such that the columns and rafters are of equal strength, that is of the same M_p value throughout the frame. Reference 2 points out that for the majority of single story structures encountered in practice, the most economical (least weight) solution will occur when the plastic moment value of the column is equal to that of the rafter. Figure 2 shows a typical single span frame and the loading which is assumed to be applied to the structure.

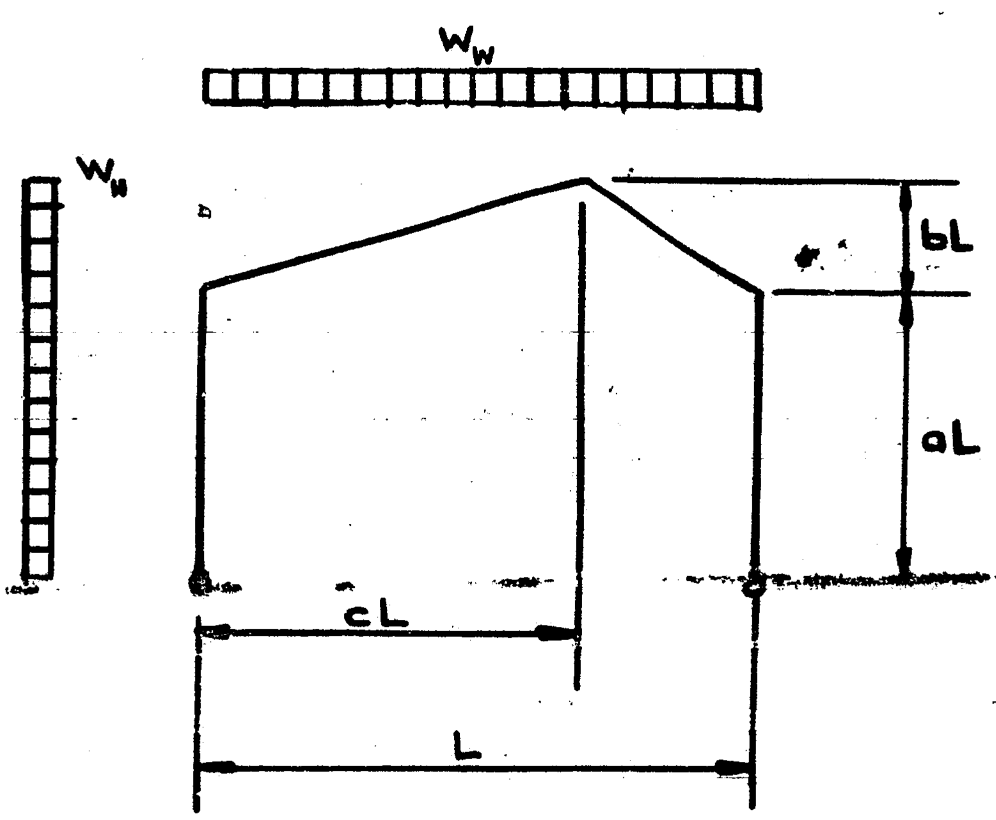

Figure 2

The analysis of the frame can be simplified, if a concentrated load is applied at the eave which has the same overturning moment about the base of the column as the uniformly distributed horizontal wind load. This simplification of the loading as shown in Figure 3, leads to slightly more conservative results. This can easily be demonstrated by assuming a mechanism which causes sidesway and comparing the amount of external work done by the two horizontal loading systems.

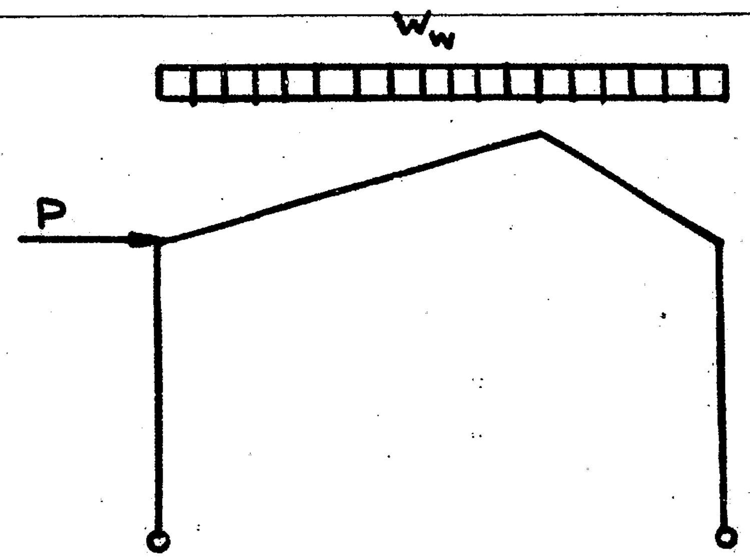

Figure 3

Plastic hinges can form at points of maximum moment or zero shear in spans loaded by distributed loads, under concentrated loads and at the ends of members meeting at a connection involving a change in geometry. Figure 4 shows the location of the possible hinge formation for the frame and loading condition shown in Figure 3.

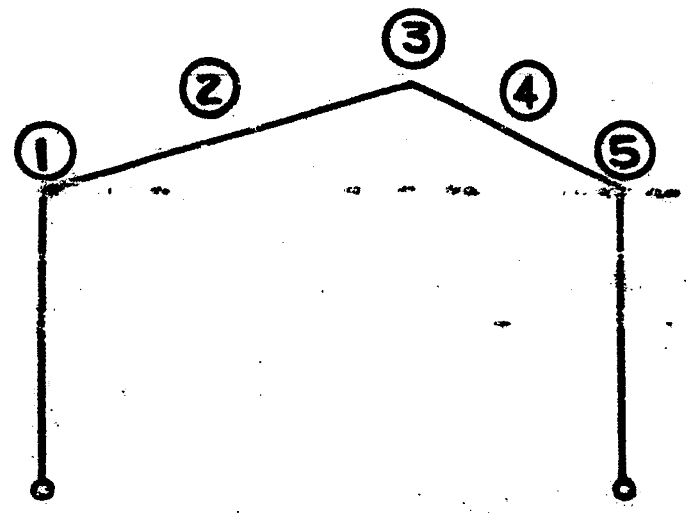

Figure 4

This frame is indeterminate to the first degree and requires the formation of two hinges to reduce the structure to a mechanism.

Once the number of possible hinge locations and the number of hinges required to reduce the frame to a mechanism has been determined, the number of possible combinations can be found from the expression for combinations (4)

$$ {}_nC_r = \frac{n(n-1)\ (n-2)\ldots.\ (n-r+1)}{r!} \qquad \ldots\ldots. \quad (1)$$

where

n = number of possible hinges

r = number of hinges to form a mechanism

C = number of combinations

For the frame shown in Fig. 4, Eq. (1) shows there are ten possible combinations of the five hinges.

$$ {}_5C_2 = \frac{(5)\ (4)}{(2)} = 10 $$

It should be pointed out for the case of the unsymmetrical frame such as the saw-tooth type, Eq. (1) will not give an indication of the number of mechanisms which require investigation, but only of the number of combination of hinges. To better illustrate this point Figure 5a shows a frame with hinges assumed to form at (1) and (3).

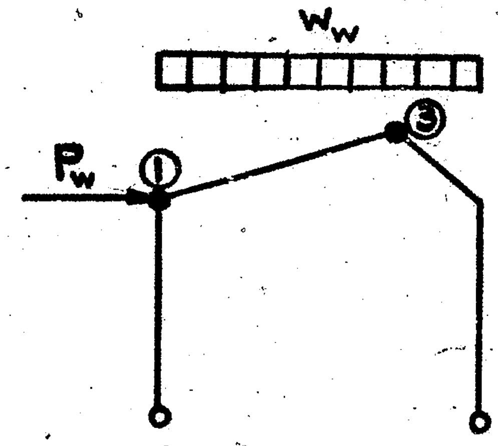

Figure 5a

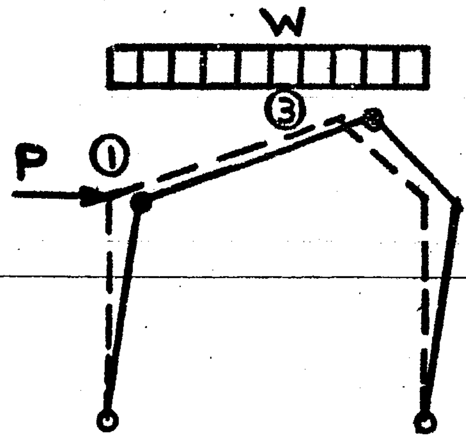

Figure 5b

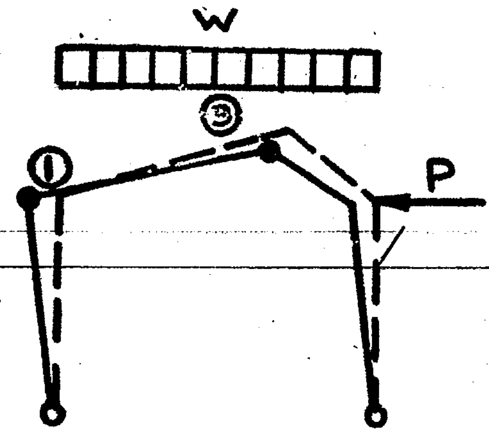

Figure 5c

Figures 5b and 5c show that the wind force may be applied to the structure from either the right or left, and in order to insure that no possible failure configurations are overlooked, for each combination of hinges, both possible sidesway conditions should be studied.

2.2 Derivation of Single Span Cases

Proceeding to determine the failure load for an assumed mechanism, the frame in Figure 6 will be analyzed. The assumed mechanism and loading is as shown, with hinges (1) and (5) forming at the eaves. The windward column is assumed to undergo an angular displacement θ.

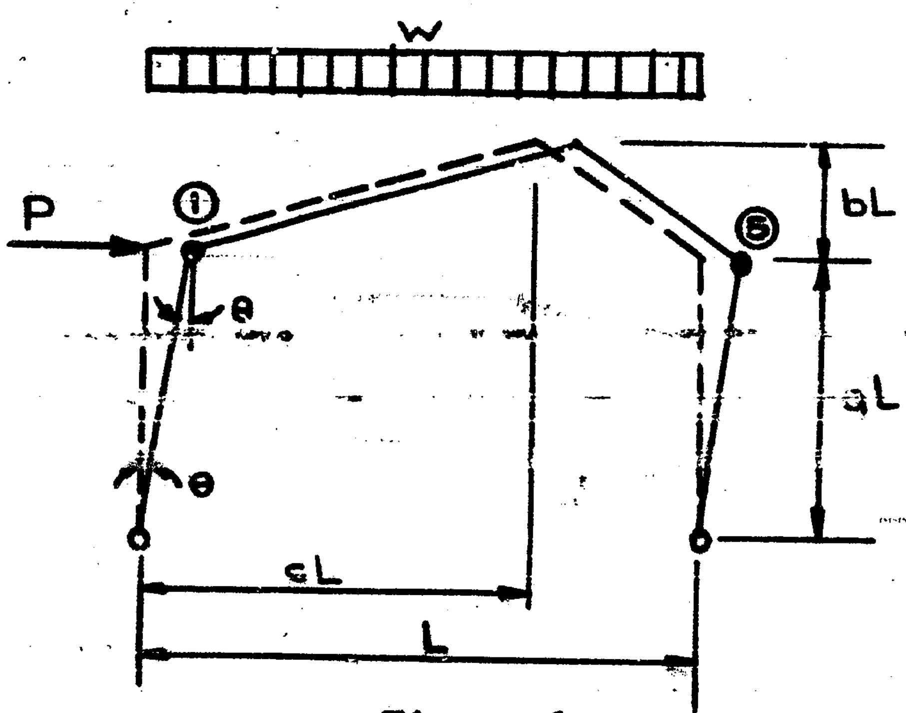

Figure 6

The Principal of Virtual Displacements states that if a system of forces in equilibrium is subjected to a virtual displacement, the total work vanishes, or the total work done by the external forces equals the total work done by the internal forces. For the assumed displacement, the rafters move as a rigid body and merely undergo a translation; therefore, the vertical load does no external work and the only external work done will be due to the horizontal displacement of the concentrated load at the eave. The expression for the external work then becomes

$$W_{ext} = PaL\theta \qquad \ldots\ldots\ldots \qquad (2)$$

Since the rafters move as a rigid body and only first order effects are considered, the distance from eave to eave must remain invariant. The horizontal displacement of the top of each column must therefore be the same and the righthand column must also rotate through the same angle θ. Each member in the frame was assumed to have the same bending resistance so the internal work expression can be shown to be

$$W_{int} = 2\ M_p\ \theta \qquad \ldots\ldots\ldots \qquad (3)$$

From the virtual displacement principal, the external and internal work expressions must be equal and the expression for the ultimate load for this assumed mechanism becomes:

$$P_{ult} = \frac{2\ M_p}{aL} \qquad \ldots\ldots\ldots \qquad (4)$$

In the next analysis, the concept of Instantaneous Centers is used of which a detailed discussion can be found in Reference 3. Figure 7 shows the frame, assumed mechanism and loading.

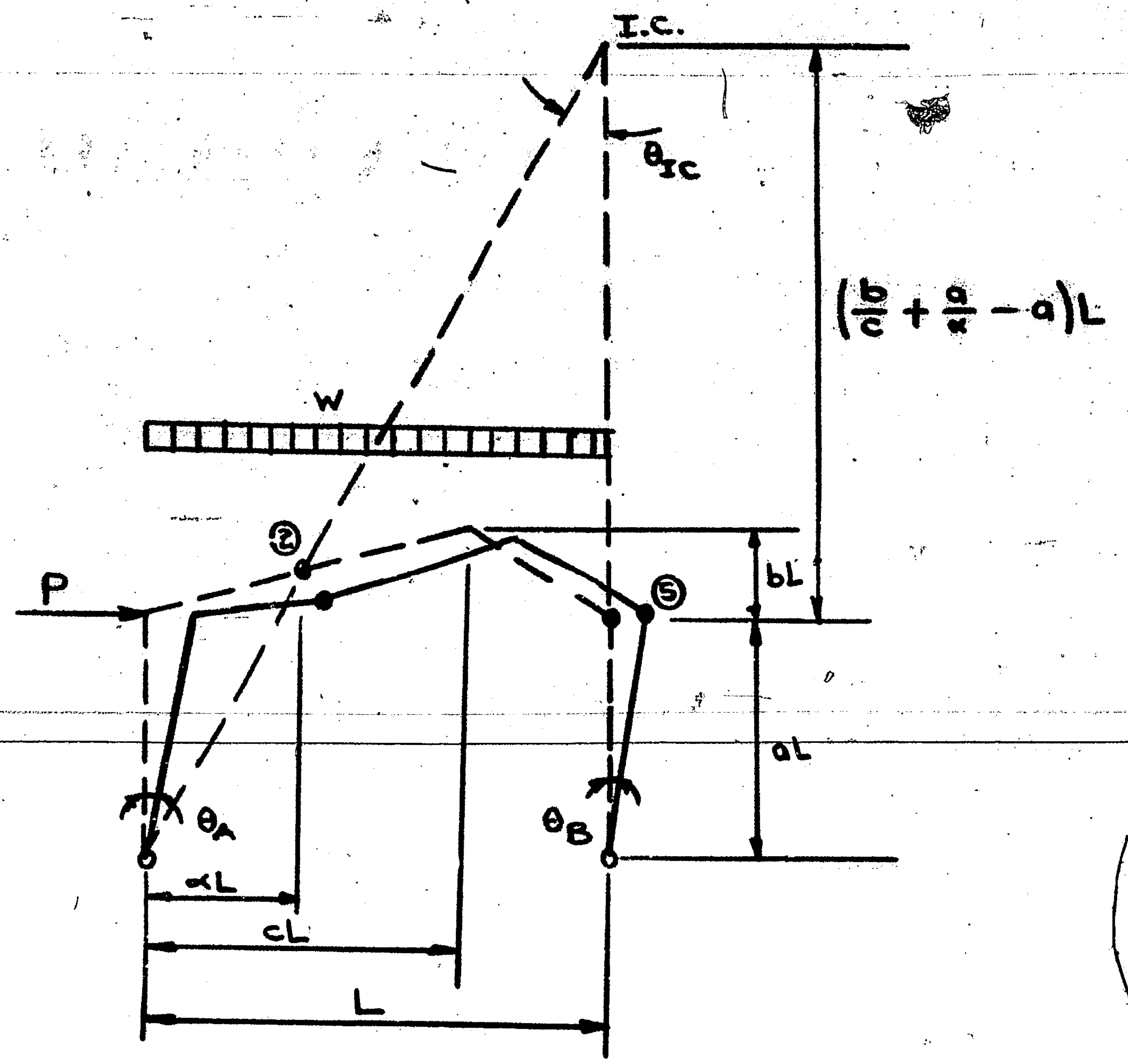

Figure 7

By assuming the distance from the lefthand column to the hinge in the windward rafter as αL, the distance from the top of the righthand column to the instantaneous center of rotation can be found to be

$$\left(\frac{b}{c} + \frac{a}{\alpha} - a\right) L$$

If the structure is given a virtual displacement to the right, all angles can readily be calculated by considering simple rigid body motion. Assuming $\theta_A = \theta$, link (2) - (5) will rotate about the I.C. with a value of

$$\theta_{I.C.} = \frac{\theta\alpha}{\left(\frac{b\alpha}{ac} + 1 - \alpha\right)} \qquad \cdots\cdots\cdot \quad (5)$$

The top of the windward column must move through the same horizontal distance as hinge (5) and it can be shown that

$$\theta_B = \frac{(1 - \alpha)\,\theta}{\left(\frac{b\alpha}{ac} + 1 - \alpha\right)} \qquad \cdots\cdots\cdot \quad (6)$$

In order to arrive at the expressions for the internal work, the virtual rotations of hinges (2) and (5) must be found. Hinge (2) will rotate through the angle θ_A plus $\theta_{I.C.}$ while (5) rotates through $\theta_{I.C.}$ plus θ_B.

$$\theta_2 = \theta \frac{1}{1-\alpha} \qquad \ldots\ldots\ldots \quad (7)$$

$$\theta_5 = \frac{\theta}{(1-\alpha)} \left(\frac{b\alpha}{ac} + 1\right) \qquad \ldots\ldots\ldots \quad (8)$$

All members are again assumed to have a plastic moment value of M_p and the internal work expression can be shown to be

$$W_{int} = M_p \left(\theta_2 + \theta_5\right)$$

or

$$W_{int} = M_p \left[\frac{\theta}{(1-\alpha)} + \frac{\theta}{(1-\alpha)}\left(\frac{b\alpha}{ac} + 1\right)\right] \qquad \ldots\ldots\ldots \quad (9)$$

External loads must be considered in parts. The external work done by the horizontal load is

$$W_{ext} = PaL\theta$$

There are several ways of arriving at the amount of external work done by the vertical load. If the area swept out in the vertical direction by the movement of the links is integrated and multiplied by the vertical load w, the result will be the expression for the external work.

All the expressions for the external work are then combined and the total external work expression will be obtained.

$$W_{ext} = \theta\left[PaL + \frac{wL^2\alpha}{2}\right] \qquad \ldots\ldots\ldots \quad (10)$$

Equating external and internal work gives

$$\frac{M_p\,\theta}{(1-\alpha)}\left[2 + \frac{b\alpha}{ac}\right] = \left[PaL + \frac{wL^2\alpha}{2}\right]\theta \qquad \ldots\ldots\ldots \quad (11)$$

It will be noticed that the external work term for vertical load contains a factor of $wL^2/2$ and the one for horizontal load does not. In the introduction, one of the assumptions was that the loads are proportional. It will now be convenient to express the wind load P in terms of the vertical load w. This can easily be accomplished if a parameter "A" is introduced which will incorporate the proportionality of the loads and the factor "a" which expresses column height as a fraction of the span L. This can be done according to the equation

$$A = 2a\left(\frac{P}{wL}\right)$$

That is,

$$P(aL) = A\left(\frac{wL^2}{2}\right) \qquad \ldots\ldots\ldots \quad (12)$$

It can also be pointed out, that if the load P at the eave is removed and the overturning moment $A\left(\frac{wL^2}{2}\right)$ is applied at the base of the lefthand column, the external work equation will remain unchanged. This procedure will be followed throughout the rest of this paper for reasons which will become obvious later.

The ultimate load for this particular mechanism can now be shown in non-dimensional form to be

$$\frac{M_p}{wL^2} = \frac{1}{2}\left[\frac{(A+\alpha)\ (1-\alpha)}{2+\frac{b}{a}\frac{\alpha}{c}}\right] \qquad \ldots\ldots\ldots \quad (13)$$

Equation 13 now contains the unknown term αL, which locates the hinge in the windward rafter. Since the structure will always fail at its first opportunity, and α is an independent variable, α can be determined by maximizing the moment expression M_p, with respect to α. The following expression can be used

$$\frac{\partial M_p}{\partial \alpha} = 0 \qquad \ldots\ldots\ldots \quad (14)$$

$$\alpha = \frac{1}{\frac{1}{2c}\frac{b}{a}}\left[\sqrt{1 - \frac{1}{2c}\frac{b}{a}\left[A\left(1 + \frac{1}{2c}\frac{b}{a}\right) - 1\right]} - 1\right] \quad \ldots\ldots\ldots \quad (15)$$

Appendix A contains all the possible collapse configurations and the derived expressions for M_p/wL^2 and α. These expressions apply to both the single span and multiple span frames. The expressions which apply to the single span case can be obtained by equating the parameter D to zero. Also contained in Appendix A is a sketch showing all the possible failure configurations. Those failure modes denoted by the letters S.S. are the governing collapse mechanisms found in constructing the sample design chart for c = 0.7.

These expressions can be used for frames of shapes other than the saw-tooth. If the b/a ratio is set equal to zero, the expressions will then be applicable to rectangular portals. By letting "c" equal 1/2 the expressions reduce to those for the symmetrical gable frame. (If "c" is given the value of 1, the expressions will describe the action of the lean-to frame.) There are certain precautions which should be taken if the expressions are to be used in the manner suggested above. That is, for every different geometry of frame there will probably be a mechanism which could not possibly occur in another shape. If the designer chooses to use the expressions of Appendix A for the analysis of a frame other than the saw-tooth variety, due to the lack of other design charts, a suitable design could be obtained by performing a moment check on the frame after the mechanism is chosen, to insure that the right choice was made.

Appendix B contains a sample design chart which has been plotted considering all possible failure configurations for the single span saw-tooth for c = 0.7. Once the geometry of the frame and loading has been decided upon, the designer need only enter the appropriate chart and the collapse mechanism and member size required can be found with very little effort. Several examples will be given later to illustrate the use of these charts.

2.3 Imaginary Frame Method

It was found that the design charts for symmetrical gable frames which are already available, can be used to design single span saw-tooth frames. The procedure used is outlined in the following example. A load factor of 1.85 is assumed when vertical load alone is acting.

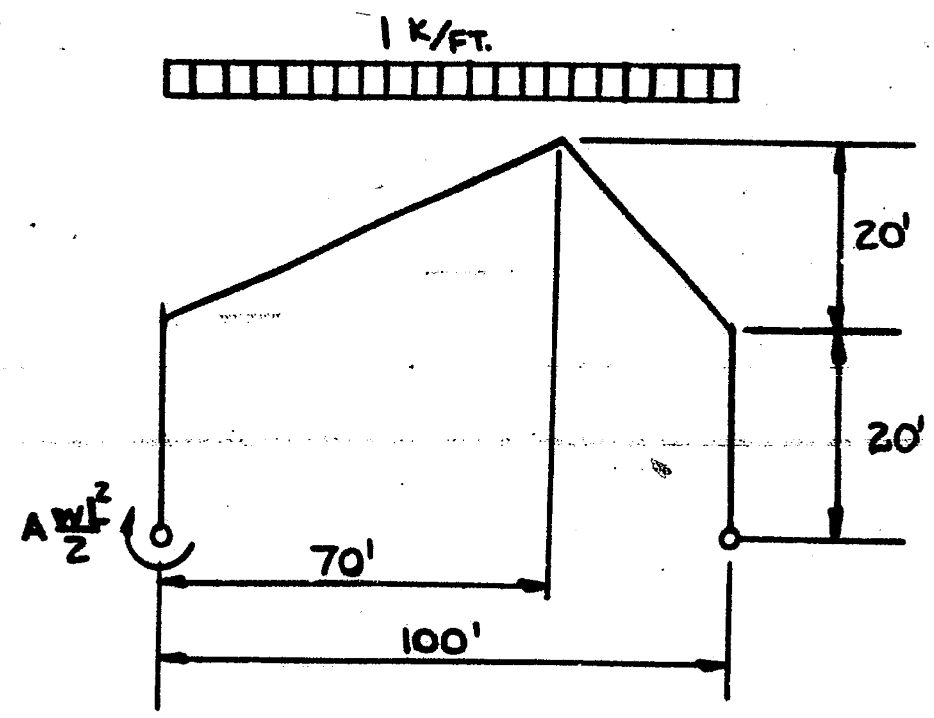

Figure 8

The following quantities can be found from the data given in Figure 8

$$\frac{b}{a} = 1.0 \qquad c = 0.7 \qquad A = 0$$

By using the appropriate design chart found in Appendix B

$$\frac{M_p}{wL^2} = 0.047$$

and

$$M_p = (0.047)\ (1)\ (1.85)\ (100)^2\ (12) = 1043 \text{ in-kips}$$

The mechanism will be that shown in Figure 9

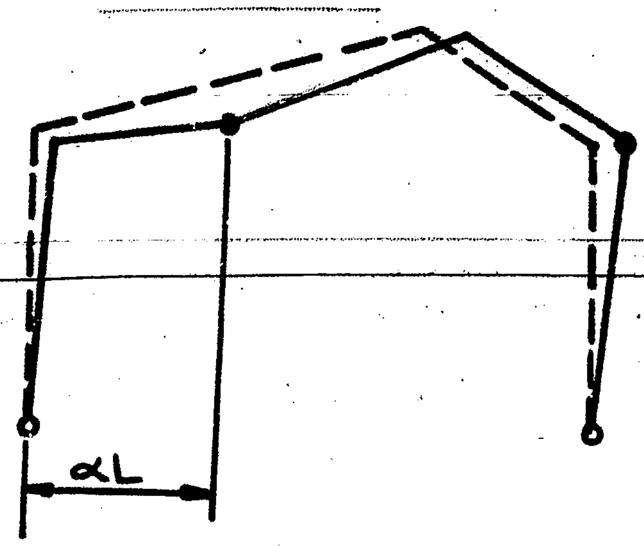

Figure 9

If an imaginary frame ABC, is constructed as shown in Figure 10, the design charts found in Reference 5 can be used

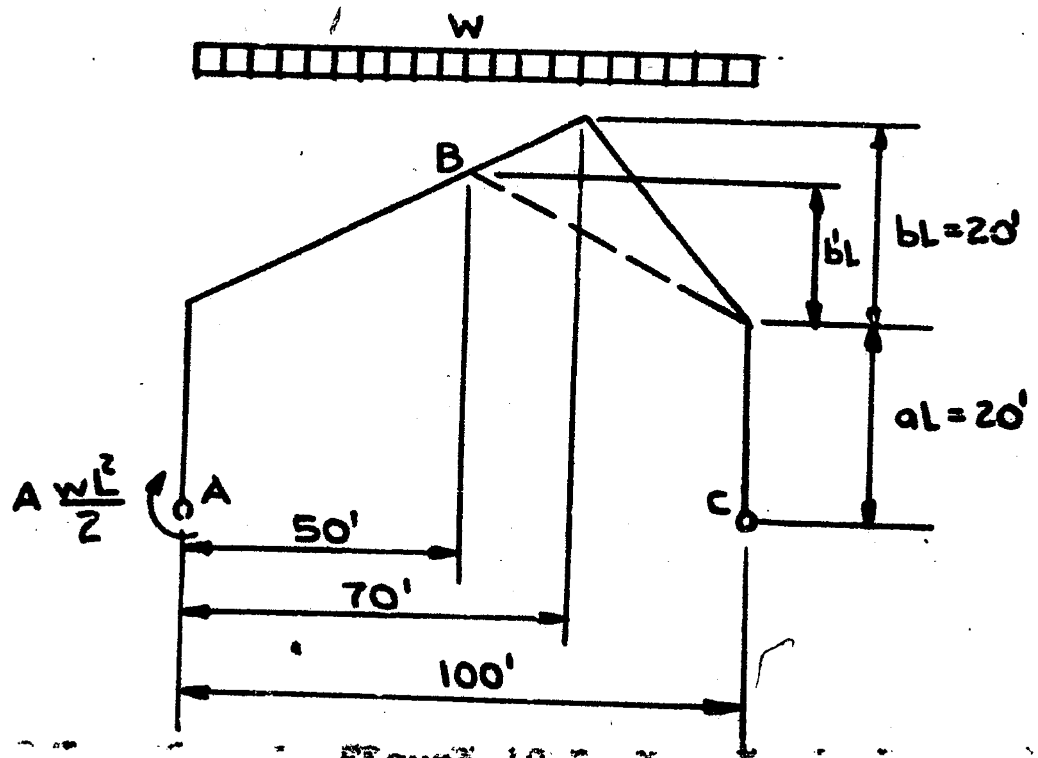

Figure 10

Figure 10 shows the following relationships for the imaginary frame.

$$\frac{b}{a} = 1.0 \qquad \frac{b'}{a} = 0.714 \qquad A = 0$$

and

$$M_p = 0.047\ (1)\ (1.85)\ (100)^2\ (12) = 1043 \text{ in-kips}$$

Therefore, it has been shown that both approaches require the same member sizes. It can be further illustrated, that both methods show the hinge location αL to be at the same position and the mode of failure in both cases, is basically the same. This imaginary frame approach works just as well when the wind loads are applied to the frame.

However, when a complete set of design curves are constructed, it may be found that this method will not be applicable to the entire range which the design charts will cover if some mechanism is found to govern which does not occur in the gable frame.

In the multiple span structure, it is possible to have α values which are greater than 0.5. This can be seen from Figure 29. Since the design charts for the gable frames show that α must have values equal to or less than 0.5 the imaginary frame method will probably not be applicable. In order to come to some conclusion as to whether the imaginary frame method is valid or not, it will be necessary to construct all the design charts for saw-tooth frames. When these charts are completed, a closer study may reveal the usefullness and limitations of this method.

III. MULTIPLE-SPAN SAW-TOOTH FRAMES

3.1 Solution by Separation

The method of instantaneous centers, used in solving the single span case, may be used in determining the ultimate load of the multiple span frame also. It is evident that as the number of spans increases and as the geometry of the frames changes, this method becomes very involved, and the algebra involved in deriving the expressions for the ultimate load becomes very complicated.

Ketter developed a method called "Solution by Separation,"(2) which as its name suggests, separates the structure under study into a number of less complicated components. The two span frame shown in Figure 11 will be used as an example.

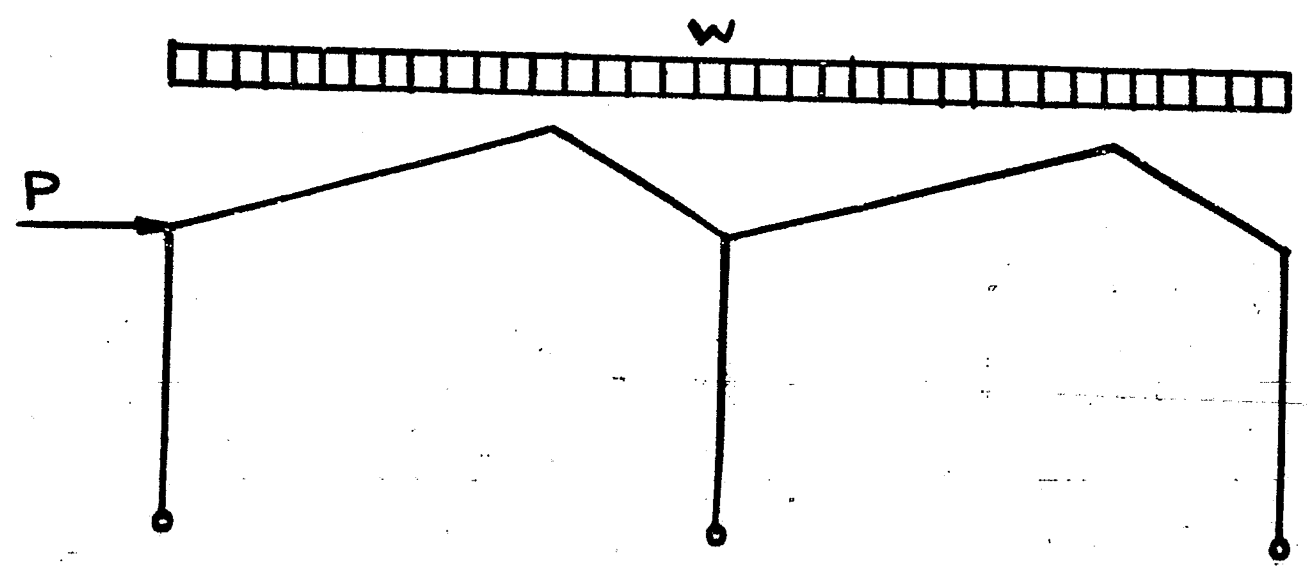

Figure 11

Reference 6 shows that the variables involved in the analysis can be separated into two groups, the loading and resistance of the righthand portion, and the loading and resistance of the lefthand portion. If the structure is divided by separating the structure at the juncture of the righthand rafter of the left frame and the center column, then the loading condition will be as shown in Figure 12.

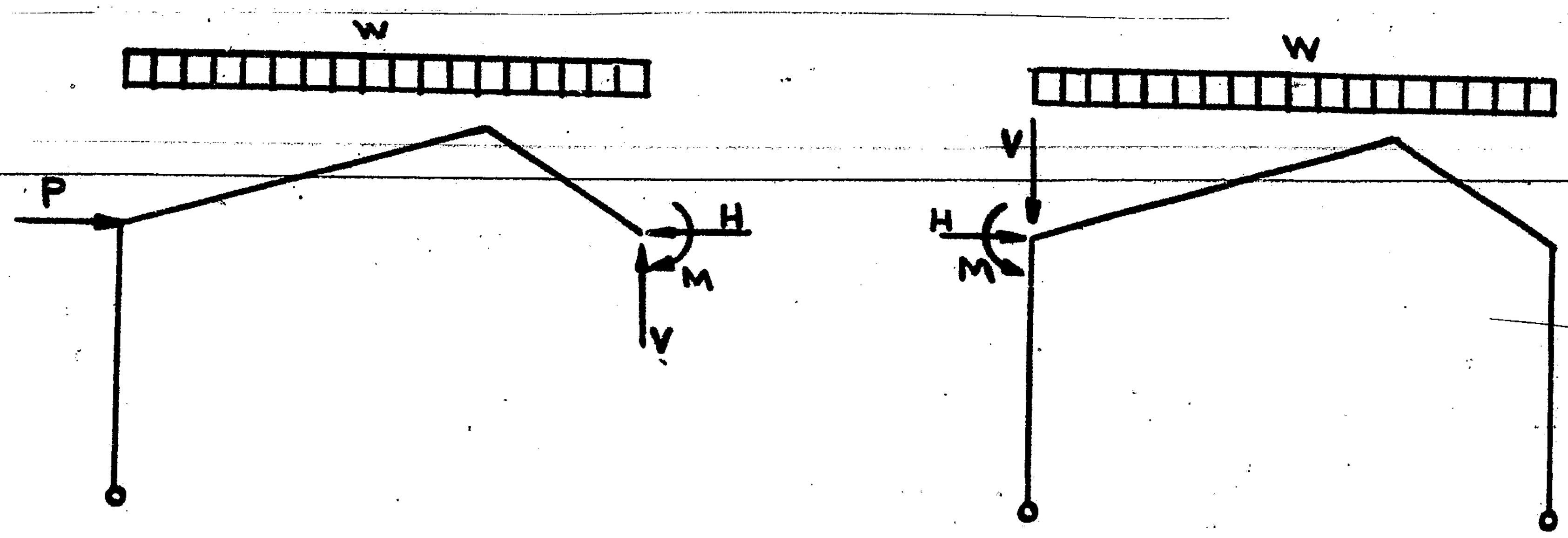

Figure 12

If each of these components are solved in terms of the loading parameters at the cut section, and then by equating the parameters the solution for the multiple span case can be obtained.

Figure 13 shows an assumed mechanism which will be used in this example.

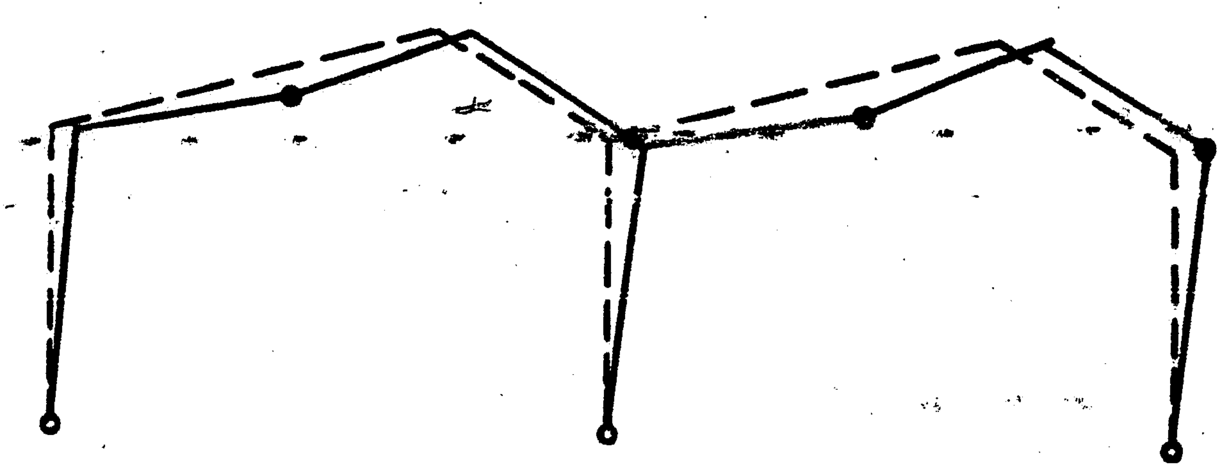

Figure 13

If the structure again is separated as in Figure 12, it can be shown that the equation for the lefthand frame reduces to

$$M_p = f\ (P,\ w,\ \alpha,\ H,\ \text{dimensions}) \qquad \ldots\ldots\ (15)$$

Similarly, the equation for the righthand structure can be found to be

$$M_p = g\ (H,\ w,\ \alpha',\ k,\ \text{dimensions})$$

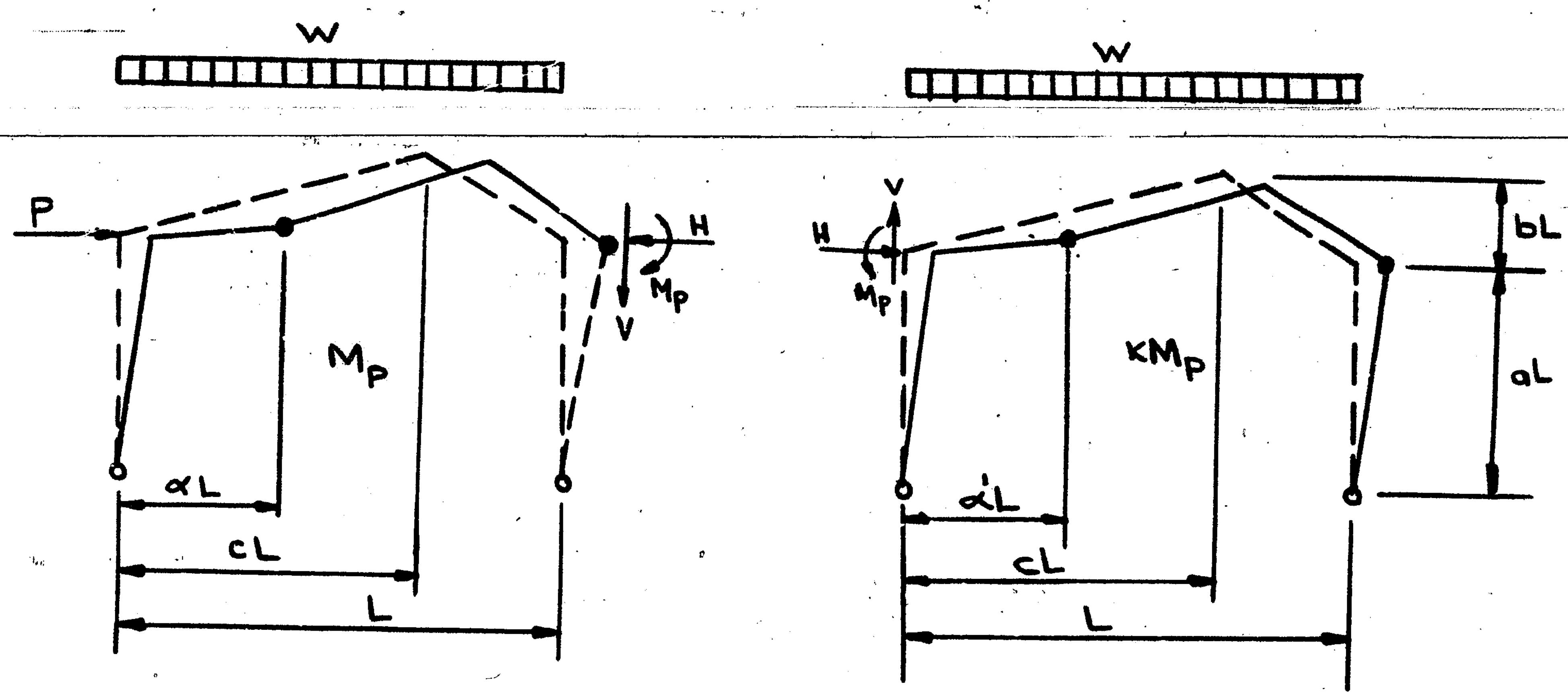

Figure 14

The shear force V at the section where the frames are separated, has no effect on the equations of external work. The work done by the horizontal force H and the moment M_p in forming the mechanism, will be equal for the two frames, since the displacements must be equal. The effects of moment and horizontal force at the point of separation, can now be grouped together into a parameter "D" and applied at the base of the column. The parameter "D" will be similar to the parameter "A" which was used earlier and the wind force P will be replaced by the parameter "A". The problem has now been reduced to that which is shown in Figure 15.

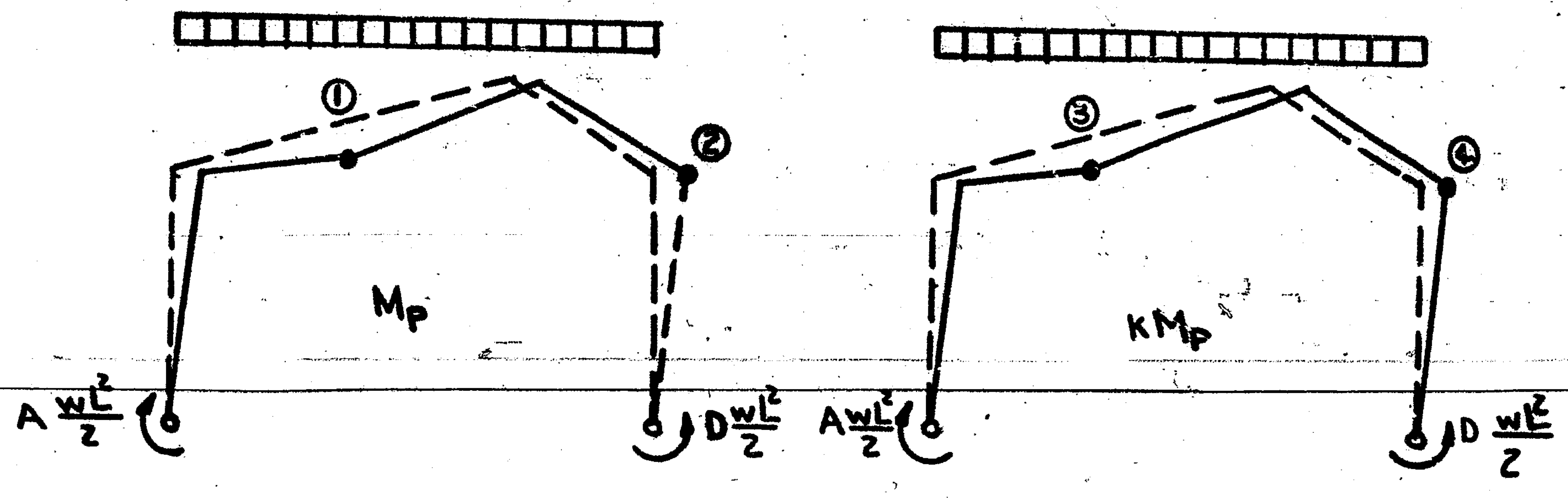

Figure 15

It should be noted at this point, that the internal work at hinge (2) is assumed to act in the lefthand sub-structure.

The structure shown in Figure 16, can now be analyzed and it will be shown that the expression obtained for the ultimate load can be easily altered to fit either sub-structure shown in Figure 15.

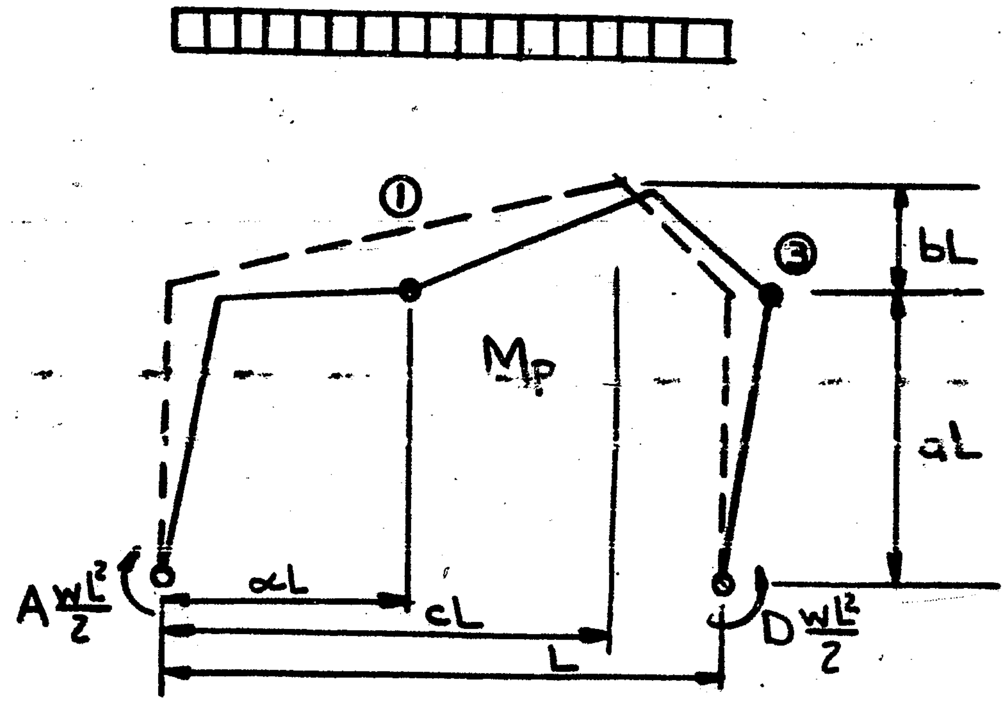

Figure 16

If the ultimate load expression is derived using the concept of instantaneous centers which was described earlier, the resulting expression would be found to be

$$\frac{M_p}{wL^2} = \frac{1}{2}\left[\frac{(A+\alpha-D)\ (1-\alpha) - D\,\frac{a}{b}\,\frac{\alpha}{c}}{2+\frac{b}{a}\,\frac{\alpha}{c}}\right] \quad \ldots\ldots\ldots \quad (17)$$

The expression for the location of hinge (1) can be found by minimizing Equation 17 with respect to α and reduces to

$$\alpha = \frac{1}{\frac{1}{2c}\,\frac{b}{a}}\left[\sqrt{1 - \frac{1}{2c}\,\frac{b}{a}\left[A\left(1+\frac{1}{2c}\,\frac{b}{a}\right) - D\left(1 - \frac{1}{2c}\,\frac{b}{a}\right) - 1\right]} - 1\right] \ldots (18)$$

If the parameter "D" is equated to zero Equation 17 reduces to

$$\frac{M_p}{wL^2} = \frac{1}{2}\left[\frac{(A+\alpha)\ (1-\alpha)}{2+\frac{b}{a}\,\frac{\alpha}{c}}\right] \quad \ldots\ldots\ldots \quad (19)$$

which is identical to Equation 13. Equation 18 will also reduce to an equation identical to Equation 15. It is now apparent that if all the possible cases are investigated with both the parameters "A" and "D" on the structure, the resulting expressions will give the ultimate carrying capacity of the multiple span frame for the assumed mechanism, and can be very easily altered to give the expression which applies for the same mechanism in the single span case or for a similar frame under a slightly different loading condition.

Appendix A contains all the expressions for the possible failure modes. The sketch shown in this section denotes by the letters M.S., those mechanisms which were found to govern in the construction of the sample design chart for the multiple span frame. Figure 28 shows a sample design chart which applies to frames contained in the multiple span structure. This chart was derived for $\alpha = 0.7$ and $\frac{b}{a} = 1.0$. Also shown on the chart are the mechanisms which will govern, once values of "A" and "D" are selected.

This chart cannot be taken as a typical example for all saw-tooth frames since the curves will obviously change, and the ranges in which certain mechanisms govern will vary as different values are selected for $\frac{b}{a}$ and c. By comparing this chart with a similar one for the gable frame it will be seen that the two charts are considerably different. This is further evidence that the "Imaginary Frame Method" mentioned earlier will probably not be applicable to the multiple span cases.

IV. DESIGN EXAMPLES

4.1 Multiple-Span Saw-Tooth Building

Two design examples will be presented to illustrate the use of these charts. The first example will be that of a three span saw-tooth frame. The building and loading shown in Figure 17 will be considered.

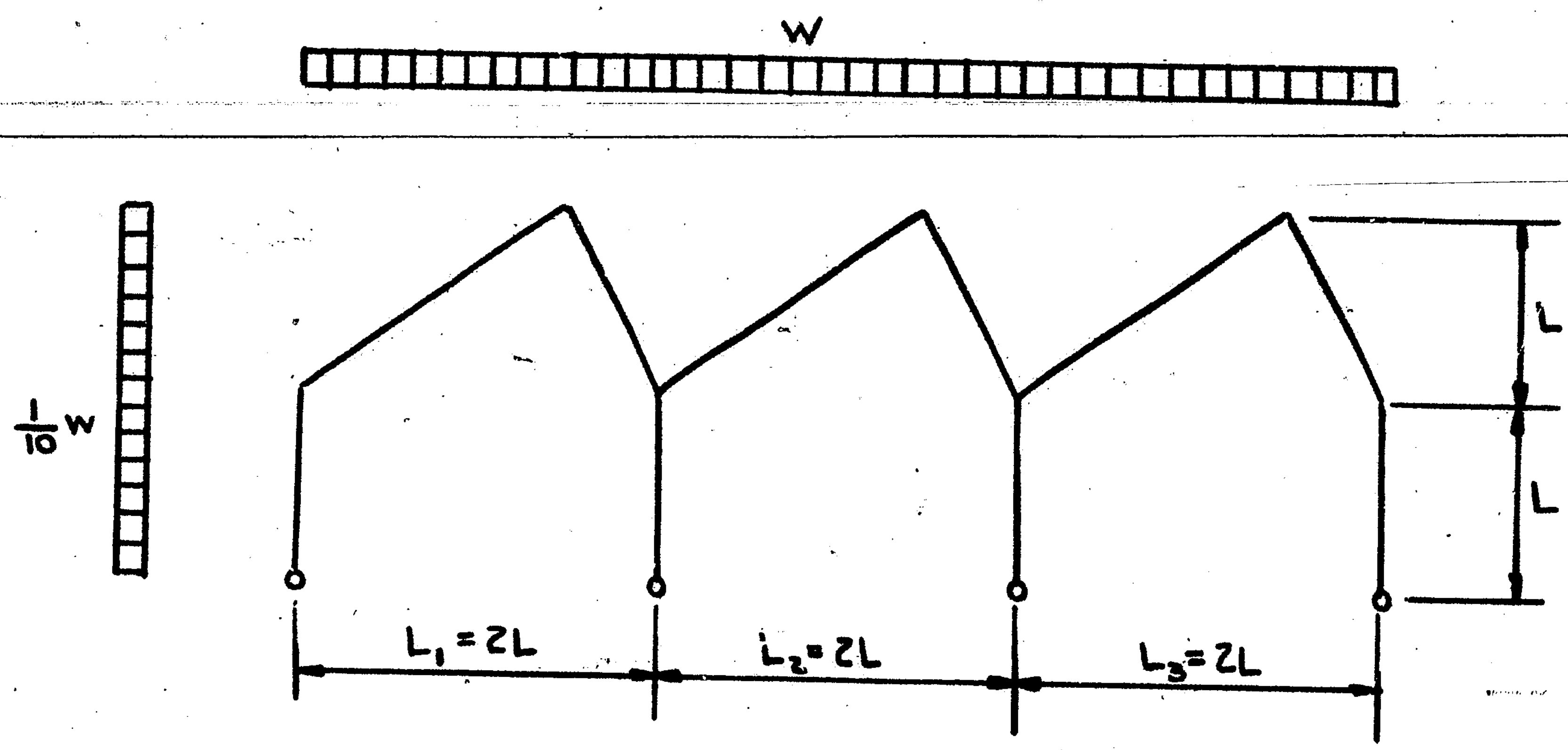

Figure 17

Spans (1), (2) and (3) b/a = 1.00

Load Factors:

Vertical Load only 1.85

Vertical Load plus wind 1.40

(A) With Vertical Load Only

Under vertical loading alone, the multiple-span frame reduces to the sub-structures shown in Figure 18.

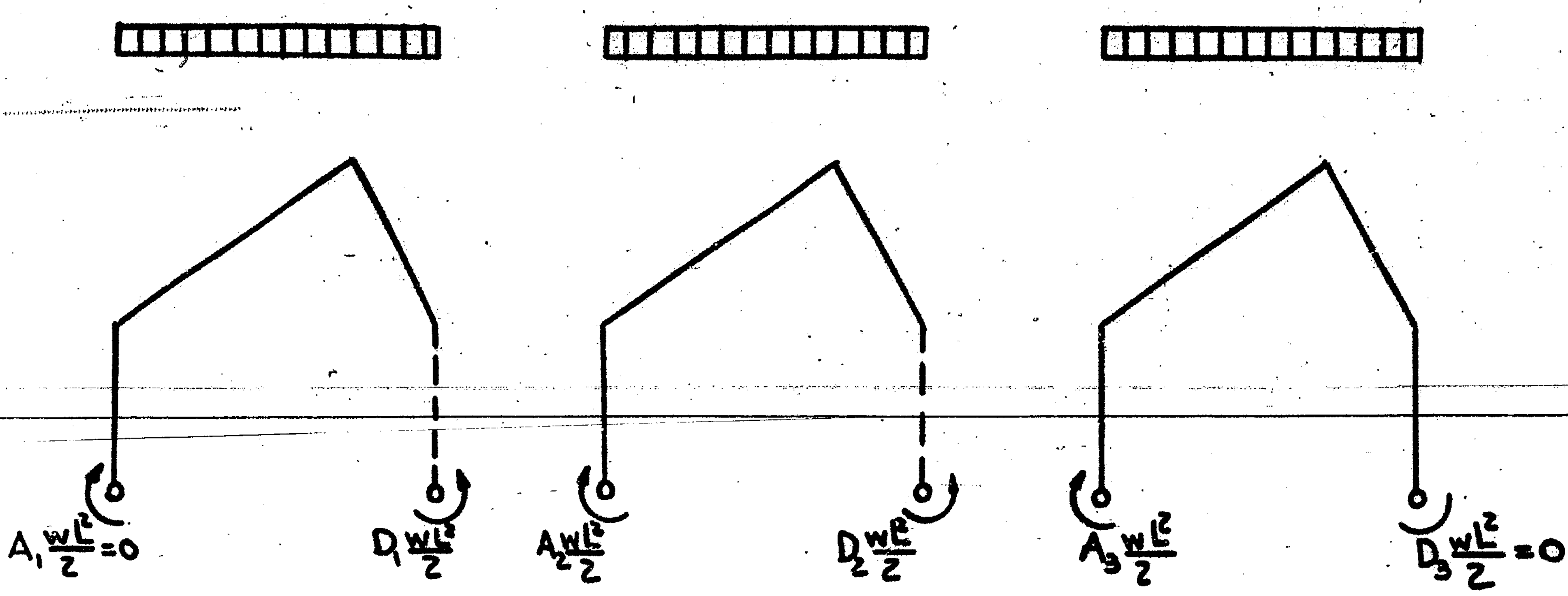

Figure 18

Due to the geometry of the frame, it will probably be more economical if all members are assumed to have the same full plastic value M_p. Since wind loads are not included in this part of the analysis $A_1 = D_3 = 0$. The geometry of all three spans being the same, also requires that

$$D_1 = A_2 \text{ and } D_2 = A_3$$

Just considering spans (1) and (3); in order for these two frames to collapse at the same load, it will be necessary that D_1 and A_3 be equal to zero. Therefore, it will also be necessary that $A_2 = D_2 = 0$. The design chart contained in Appendix B then shows the resulting solution to be:

$$\frac{M_p}{w_w L_1^2} = 0.0466$$

or when 2L is substituted for L_1

$$\frac{M_p}{w_w L^2} = 0.1864$$

(B) Vertical Load Plus Wind From The Left

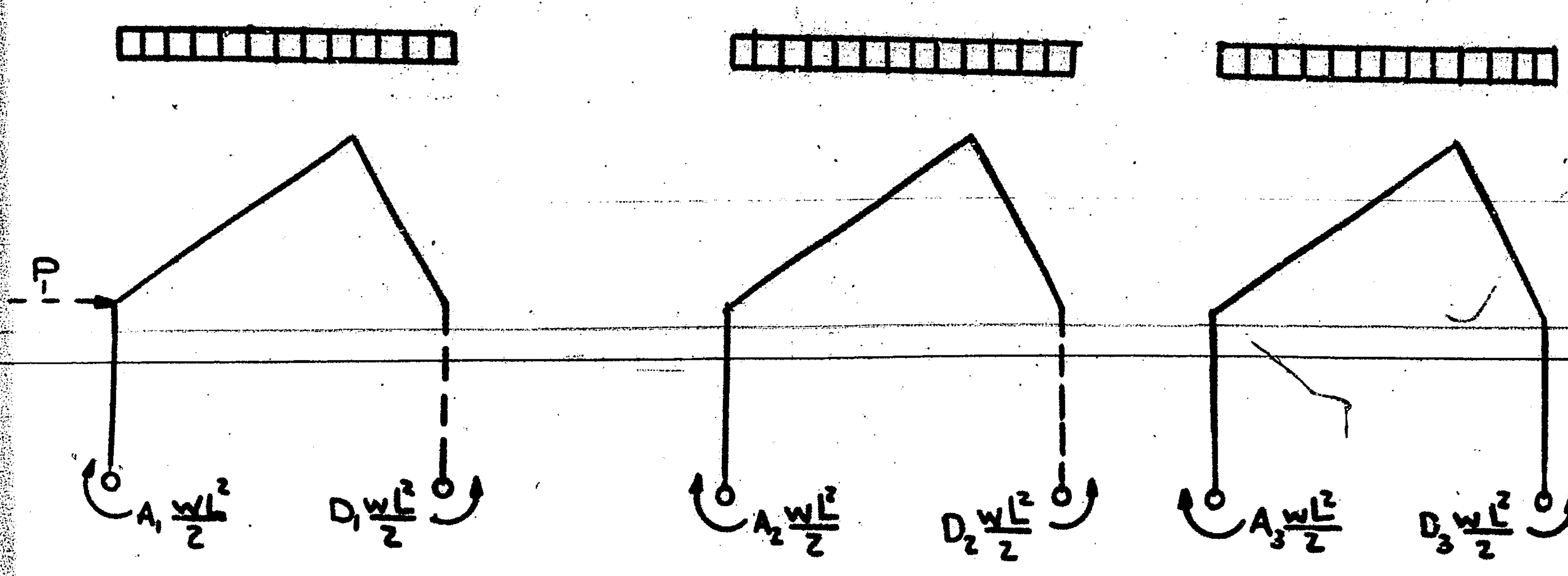

Figure 19

In Figure 19, the sub-structures of the frame are shown with the horizontal wind load replaced by a concentrated load and this concentrated load replaced by its overturning moment about the base, $A_1 \frac{wL^2}{2}$. From conditions stated earlier, the following relationships can also be found.

$$D_2 = A_3 \qquad D_1 = A_2$$

The loading parameter A_1 is then determined

$$P_1 L = A_1 \frac{4wL^2}{2} = \frac{1}{10} w(2L)L$$

$$A_1 = 0.1$$

The following table car now be constructed by assuming values for D_1

TABLE 1

(1) A_1	(2) D_1	(3) M_p/w_wL^2	(4) $A_2 = D_1$	(5) M_p/w_wL^2	(6) $D_2 = A_3$	(7) D_3	(8) M_p/wL^2
0.1	0.050	0.0470	0.050	0.0470	0.0219	0	0.0490
0.1	.045	.0475	.045	.0475	.0176	0	.0485
0.1	.043	.0480	.043	.0480	.0150	0	.0480

It was stated earlier that all members in the structure would have the same M_p. The problem was approached by assigning a value to D_1, thus enabling the designer to determine values for D_2 and A_3. Values of D_1 were adjusted until columns (3), (5) and (8) in Table 1 gave the same value of M_p/wL^2.

(C) Governing Case

Comparing the analysis for vertical load alone with the one which includes wind loading, the controlling condition and member size can be determined.

(A) Vertical Load Alone (including load factor)

$$\frac{M_p}{wL^2} = 0.3448$$

(B) Vertical Load Plus Wind From the Left (including load factor)

$$\frac{M_p}{wL^2} = 0.2690$$

Therefore, it has been shown that for the geometry and loading assumed, the vertical load alone will control the design.

The case in which vertical load and wind from the right are acting, is not considered since it will obviously require a higher load to form a mechanism.

4.2 Mill Building

The mill building shown in Figure 20 will now be considered as a second design example. The following relationships can be

Span L_1	$b/a = 1.0$	$L_1 = 2L$
Span L_2	$b/a = 0.2$	$L_2 = 4L$
Span L_3	$b/a = 1.0$	$L_3 = 2L$

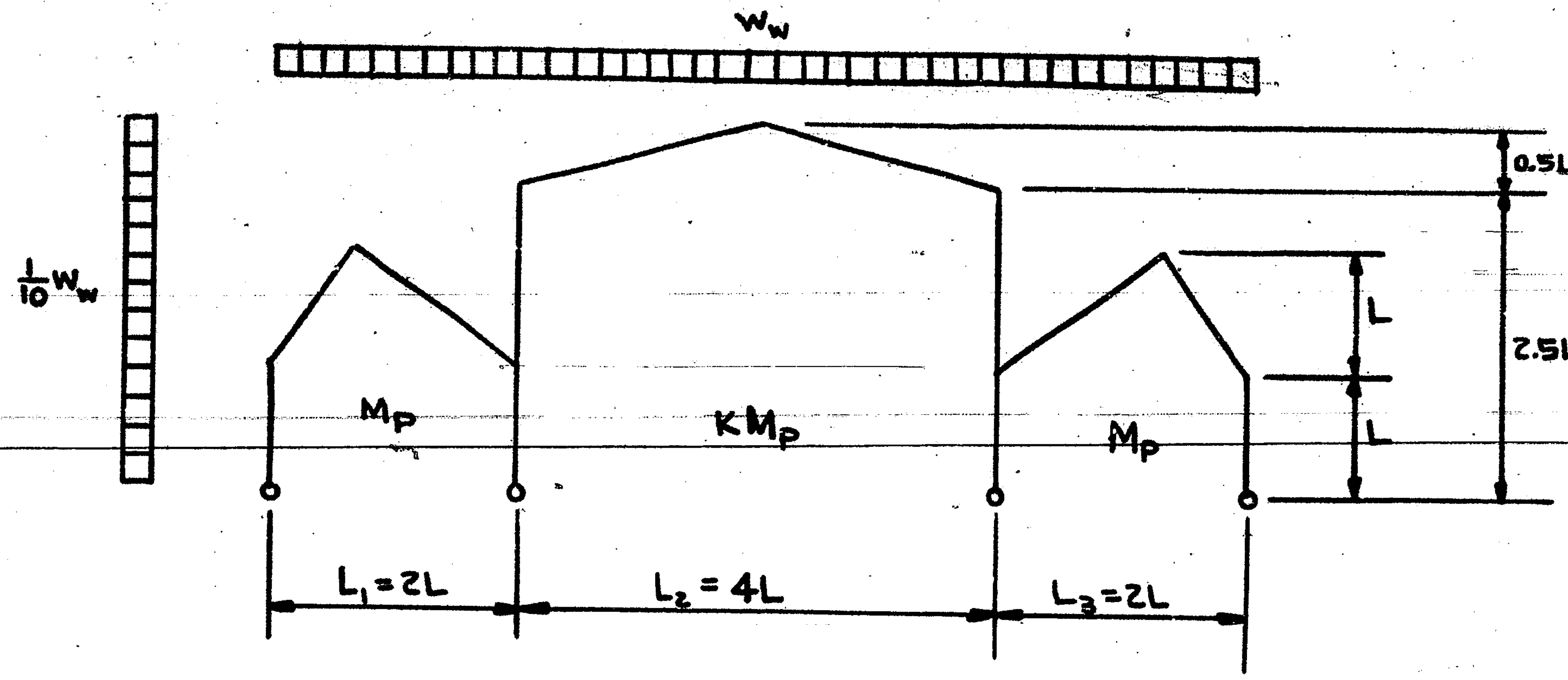

Figure 20

(A) Vertical Load Only

The case in which vertical load alone acts will first be considered using the same load factors as were used in the previous example. The structure reduces to the three sub-structures shown in Figure 21.

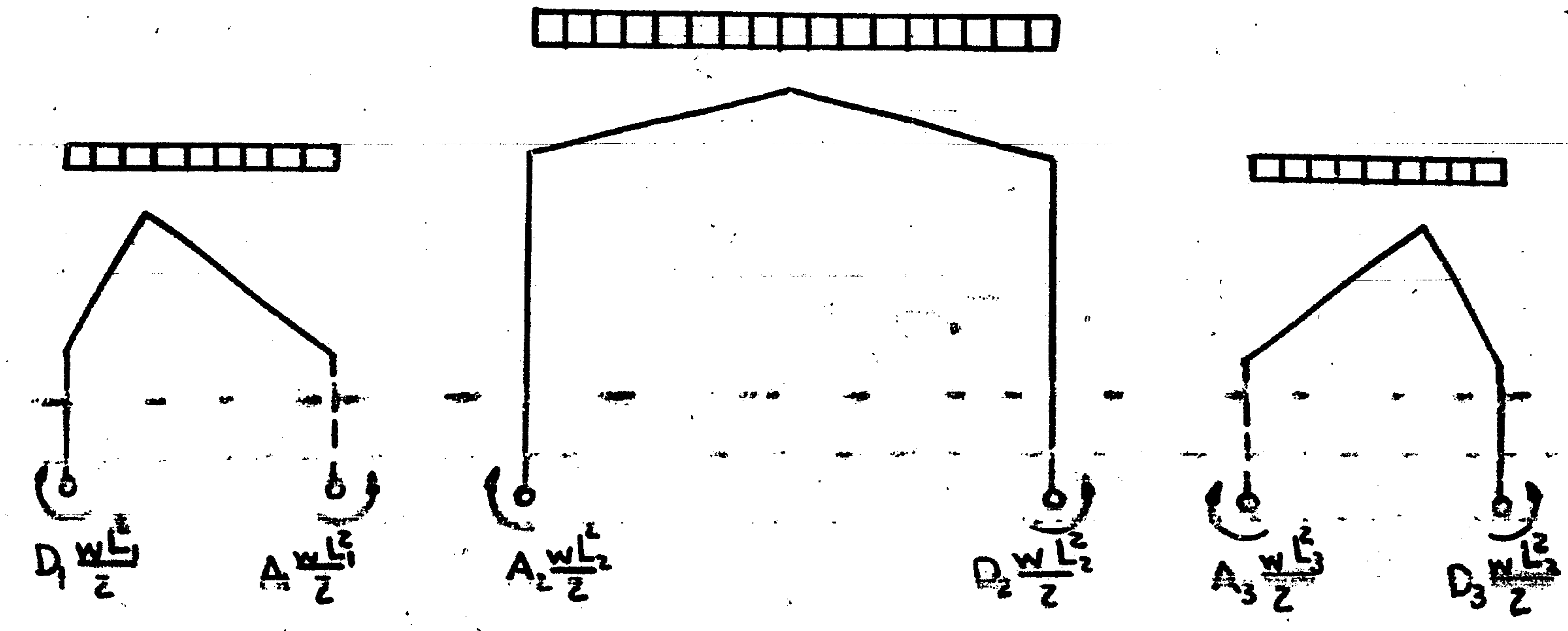

Figure 21

By considering the loading and geometry, the following relationships can be found:

$$D_1 = 0 \qquad D_3 = 0$$

$$A_1 \frac{4wL^2}{2} = A_2 \frac{16wL^2}{2} \qquad A_1 = 4A_2$$

$$D_2 \frac{16wL^2}{2} = A_3 \frac{4wL^2}{2} \qquad A_3 = 4D_2$$

Due to the symmetry of the structure and the loading

$$A_2 = D_2 \text{ and } A_1 = A_3$$

By assuming values for A_1 or A_2, the following table of relationships can be found by using the appropriate design charts.

TABLE 2

A_1	$M_p/w_wL_1^{2*}$	$A_2 = D_2 = \frac{A_1}{4}$	$KM_p/w_wL_2^{2**}$	M_p/wL^2	KM_p/wL^2
0	0.0466	0	0.0566	0.3448	1.6754
0.100	.0584	0.025	.0560	.4322	1.6576
0.200	.0720	.050	.0548	.5328	1.6221
0.300	.0878	.075	.0539	.6497	1.5954
0.400	.1058	.100	.0524	.7829	1.5510

* See Figure 28

** These values determined from Reference (7)

(B) Wind Load and Vertical Load Acting

The sub-structures are shown in Figure 22 for the case where both wind and vertical load are acting. Due to the symmetry of the frame and the loading, only wind from one side need be considered.

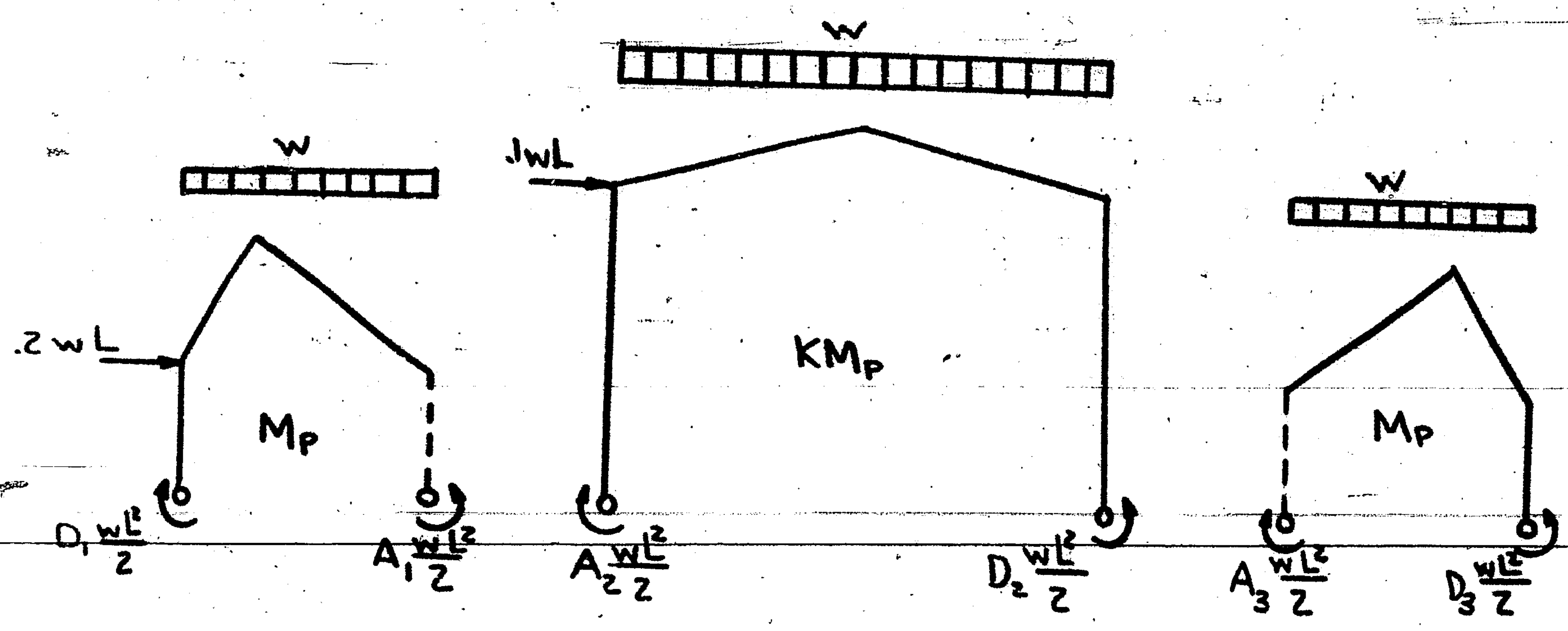

Figure 22

The frame shown in Figure 22 can be designed for two different conditions. The saw-tooth frames can be designed such that KM_p is smaller than M_p or they could be designed such that K is greater than one. Both conditions will be considered.

From the information shown in Figure 22, the following expressions can be derived:

For Span (1)

$$D_1 \frac{wL_1^2}{2} = (.2wL)\, L \qquad D_1 = 0.1$$

Between Spans (1) and (2)

$$A_2 \frac{wL_2^2}{2} = A_1 \frac{wL_1^2}{2} + (0.1wL)\,(2.5L)$$

or

$$A_2 = 0.25A_1 + 0.0313$$

Between Spans (2) and (3)

$$D_2 \frac{wL_2^2}{2} = A_3 \frac{wL_3^2}{2}$$

or

$$D_2 = 0.25A_3$$

For Span (3)

$$D_3 = 0$$

(a) Side Spans the Smallest (K>1)

So that both side spans require the same M_p for collapse, the design chart contained in Figure 28 will give the following results:

$$D_1 = 0.1 \qquad A_1 = 0.187$$

$$\frac{M_p}{w_w L_2^2} = 0.0466$$

From the relationships previously derived

Span (3)

$$D_3 = 0 \qquad A_3 = 0.$$

and

$$\frac{M_p}{w_w L_3^2} = 0.0466$$

Span (2)

$$D_2 = 0 \qquad A_2 = 0.078$$

$$\frac{KM_p}{wL_2^2} = 0.0664$$

Rewriting the terms to contain the load factor 1.40 for wind loading and putting the expressions in terms of the span parameter L

$$\frac{M_p}{wL^2} = 0.2610$$

$$\frac{KM_p}{wL^2} = 1.4874$$

(b) Center Span the Smallest

Figure 23 shows the information required to analyze this case. Due to the unsymmetrical loading and the geometry of the frame, it can be shown that there will be a number of possible solutions. Reference 5 shows that by equating A_2 and D_2, the center span will have its smallest possible member size. It will not be possible to adjust the outside spans to have the same plastic moment value, but for this the span on the right will require a larger member size than the one on the left.

The relationships derived in the previous case will be used in addition to the condition that $A_2 = D_2$

Therefore

$$D_1 = 0.1 \qquad D_3 = 0 \qquad A_2 = D_2$$

$$A_1 = A_3 - .125$$

$$A_2 = D_2 = 0.25A_1 + 0.0313$$

Values are now assigned to A_3 and Table 3 can be constructed.

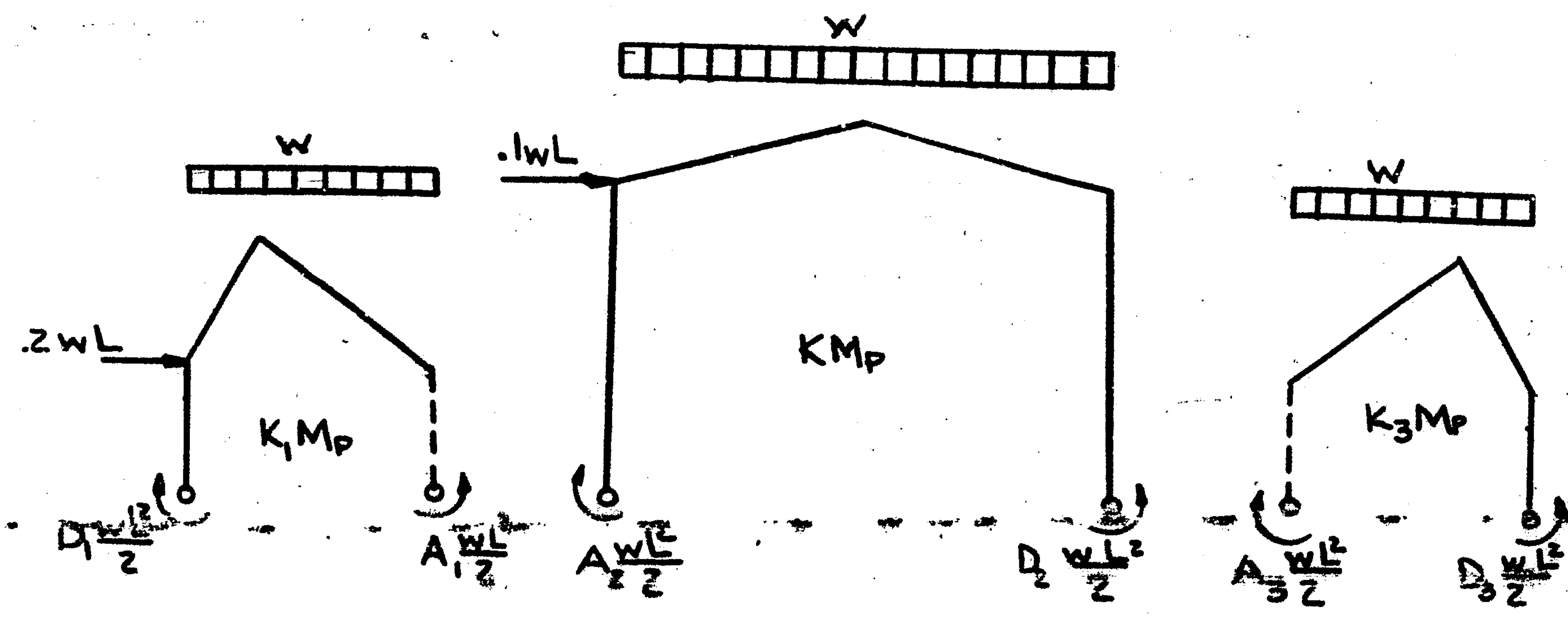

Figure 23

TABLE 3

D_3	A_3	$K_3\frac{M_p}{w_wL_3{}^2}$	A_1	D_1	$K_1\frac{M_p}{w_wL_1{}^2}$	A_2	$K\frac{M_p}{w_wL_2{}^2}$	$K_3\frac{M_p}{w_wL_i{}^2}$	$K_1\frac{M_p}{w_wL^2}$	$K\frac{M_p}{w_wL^2}$
0	0.2	0.0720	0.075	0.1	0.0320	0.050	0.0550	0.2880	0.1280	0.8800
0	0.3	.0875	.175	0.1	.0450	.075	.0540	.3500	.1800	.8640
0	0.4	.1055	.275	0.1	.0601	.100	.0524	.4220	.2404	.8384
0	0.125	.0613	0	0.1	.0305	.0313	.0568	.2452	.1220	.9088
0	0.150	.0648	.025	0.1	.0305	.0375	.0556	.2592	.1220	.8896
0	0.175	.0682	.050	0.1	.0305	.0437	.0552	.2728	.1220	.8832

If the $\frac{K_iM_p}{w_wL^2}$ values which are listed in Table 3 are multiplied by the total length of member (L_i) in each respective frame and the summation of relative weights for each assumed A_3 value are plotted as shown in Figure 24, the optimum value for A_3 can be found. (See Table 4 and Figure 24).

TABLE 4

A_3	$\frac{K_1M_p}{wL^2}L'_1$	$\frac{K_3M_p}{wL^2}L'_3$	$\frac{KM_p}{wL^2}L'_2$	$\sum K_iM_pL'_i$
0.125	0.474	0.953	9.200	10.627
0.2	.497	1.119	8.909	10.525
0.3	.699	1.360	8.747	10.806
0.4	.934	1.640	8.488	11.062
0.150	.474	1.007	9.006	10.487
0.175	.474	1.060	8.942	10.476

$L'_1 = L'_3 = 3.886L$

$L'_2 = 10.124L$

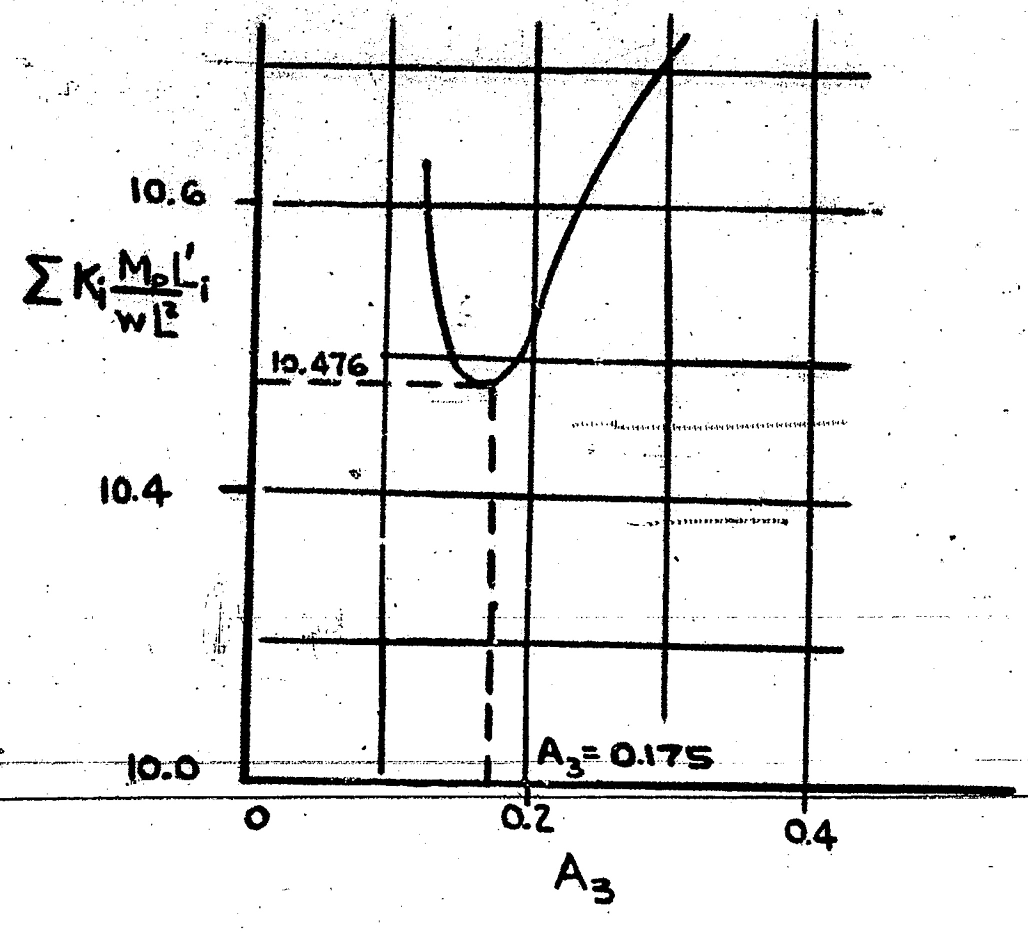

Figure 24

By introducing the load factor for combined wind and vertical load, the following values are obtained

$$K_1 \frac{M_p}{wL^2} = 0.1708$$

$$K \frac{M_p}{wL^2} = 1.2365$$

$$K_3 \frac{M_p}{wL^2} = 0.3819$$

Since wind was only considered acting from the right, it is apparent that both side spans would have to be assigned the largest value $K_i \frac{M_p}{wL^2}$ determined for the exterior spans to cover the case of wind acting from the left. The factor K_3 is also assigned the value of one so that a comparison can be carried out with the other possible conditions which were investigated. The above equations now reduce to

$$K \frac{M_p}{wL^2} = 1.2365$$

$$\frac{M_p}{wL^2} = 0.3819$$

(C) Governing Case

The analyses which were carried out assuming that the center span was as small as possible and the end spans as small as possible, defined the two extremes for the wind loading. A plot as shown in Figure 25, can now be made from the information contained in Table 2 for vertical load alone and that which was found in the preceeding analysis for the wind plus vertical load cases.

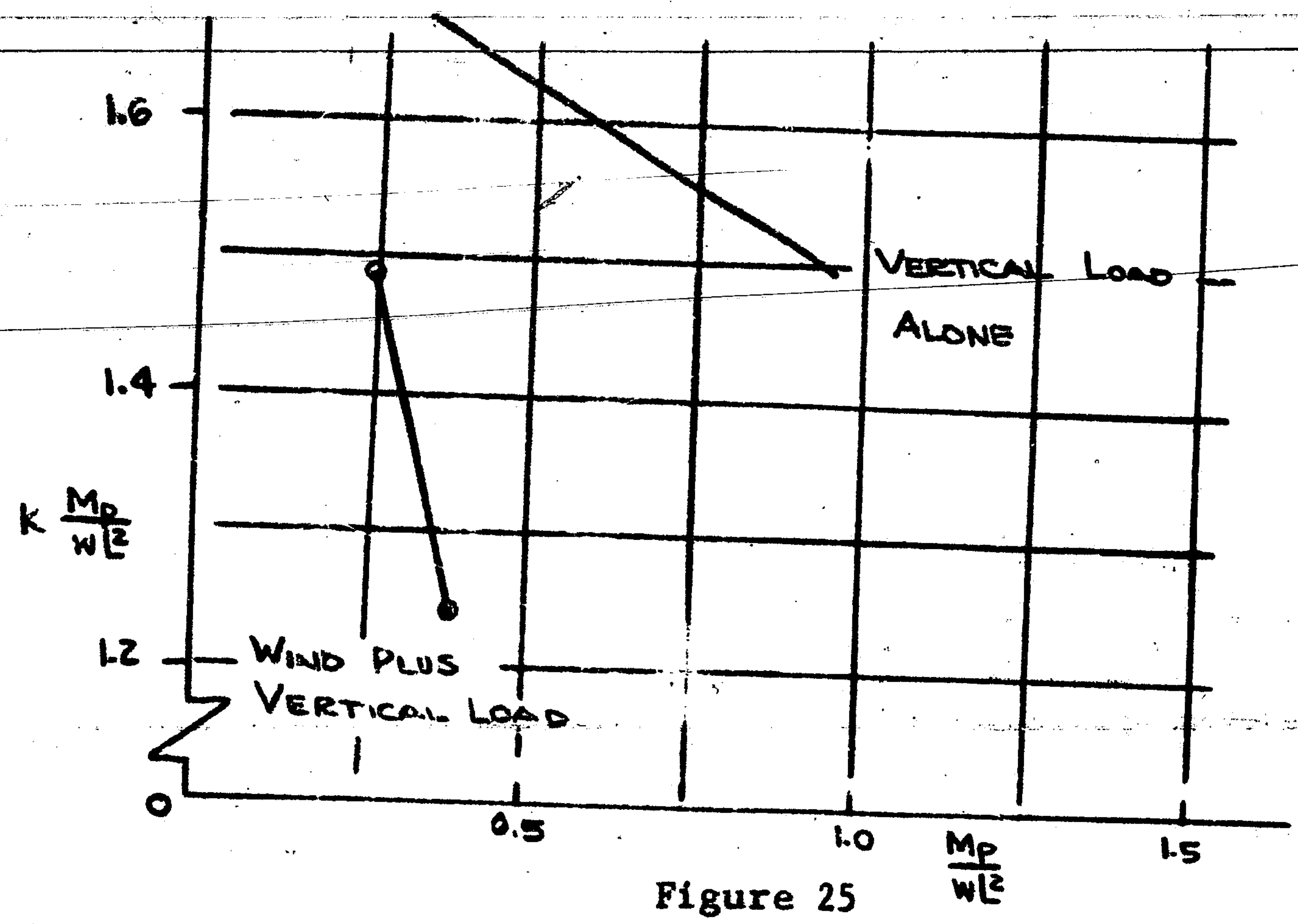

Figure 25

The two curves shown in Figure 25 are approximated by straight lines, although the solutions will not be linear functions they come very close to being linear and this assumption will not affect the results in this particular case. The structure in question must provide the greatest M_p and KM_p and therefore, the case where vertical load alone acts will govern the design.

(D) Least Weight Design

Figure 25 shows there are many solutions to the case when vertical load alone acts. Each solution will result in a different choice of relative member size and that choice which gives the least total weight to the structure will be considered the most desirable solution.

Assuming a linear relationship between the fully plastic moment value M_p, and the unit weight W,

$$W = CM_p$$

the total weight of the frame can be determined from the relationship

$$\text{TOTAL WEIGHT} = \sum_{i=1}^{n} (W L_i) = C\sum_{i=1}^{n} (M_{pi} L_i)$$

The relative total weight, is all that is important in determining the combination of member sizes which result in a least weight solution, a weight function in terms of only M_{pi} and L_i can be used.

$$\text{WEIGHT FUNCTION} = \frac{\sum_{i=1}^{n} (WL_i)}{C} = \sum_{i=1}^{n} (M_{pi} L_1)$$

It will be assumed that the interior columns will have the same member size as the rafters in the interior span. The following weight function can then be derived for this particular mill building

$$\rho = 7.772L\ (M_p) + 10.124L\ (KM_p)$$

Reducing the above equation to non-dimensional form

$$\frac{\rho}{wL^3} = 7.772 \left(\frac{M_p}{wL^2}\right) + 10.124\left(\frac{KM_p}{wL^2}\right)$$

The results from Table 2 are then used in combination with the above equation to find the information given in Table 5.

TABLE 5

$\frac{M_p*}{wL^2}$	$\frac{KM_p*}{wL^2}$	$7.772\ \frac{M_p}{wL^2}$	$10.124\ \frac{KM_p}{wL^2}$	$\frac{\rho}{wL^3}$
0.3448	1.6754	2.662	16.962	19.624
.4322	1.6576	3.337	16.782	20.119
.5328	1.6221	4.113	16.422	20.535
.6497	1.5954	5.016	16.152	21.168
.7829	1.5510	6.044	15.702	21.746

* From Table 2

The solution which gives the smallest value for $\frac{\rho}{wL^3}$ is then the least weight solution. For the mill building and loading considered in this example, the least weight solution is then (including load factor and span parameters)

$$\frac{M_p}{wL^2} = 0.345$$

and

$$\frac{KM_p}{wL^2} = 1.675$$

V. SUMMARY

Expressions necessary to develop design charts for single and multiple span saw-tooth frames were derived. These expressions consider all possible modes of failure for frames loaded by dead load alone or frames loaded by a combination of wind and dead loading. The expressions derived can also be applied to frames of other geometry by slight modifications. For example: if c is assigned the value of one, the equations apply to the lean-to frame; if c = equals 1/2 the equations for the gable frame can be derived; if b/a is set equal to zero, the expressions reduce to those of the rectangular portal frame. Care should be used in applying these equations to frames of other geometry since there always exists a possibility that some special mechanism which could not occur for the saw-tooth variety will govern for another geometry.

In this paper, a method developed by Ketter was used to construct a sample design chart for both the single span and multiple span saw-tooth frames.

A possible method of designing single span saw-tooth frames by using the charts already available for the symmetrical gable frame is discussed in the section called "Imaginary Frame Method." In order to find the limitations to the applicability of this method, it will require further study.

The design examples include a three span saw-tooth frame which was analyzed for the given loading conditions and the resulting member sizes determined. A mill type building was also analyzed to illustrate how the charts for the saw-tooth frame may be used in conjunction with other charts which are already available.

VI. REFERENCES

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5. Ketter, R.L., "Plastic Design Of Pinned-Base Gable Frames", Welding Research Council Bulletin No. 48, New York, (March 1959).

6. Ketter, R.L., Yen, B.T., "Plastic Design Of Pinned-Base Lean-To Frames", Welding Research Council Bulletin, No. 53 , New York, (September 1959).

7. "Plastic Design In Steel", American Institute of Steel Construction, New York, (1959).

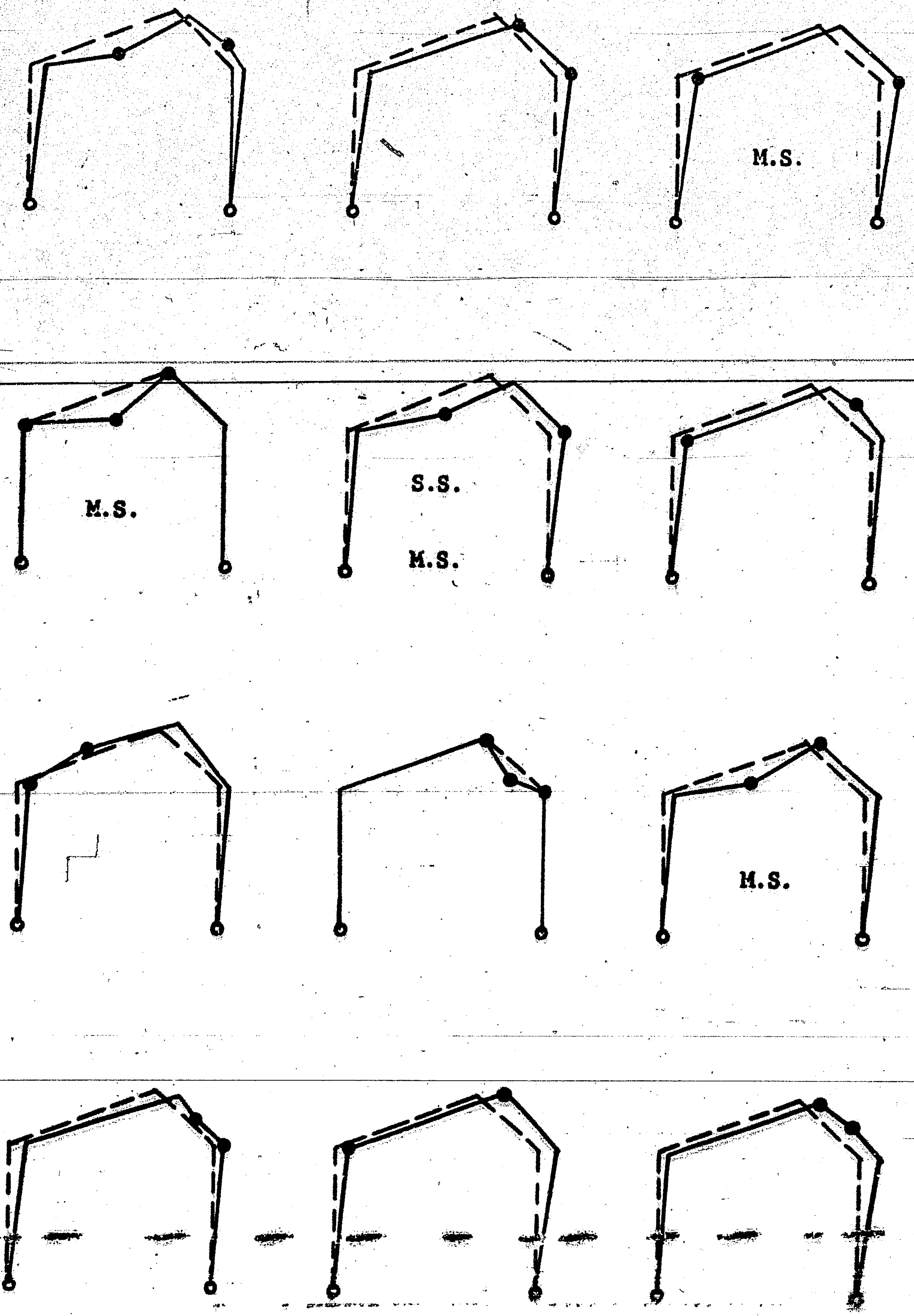
VII. NOMENCLATURE

a	=	nondimensional parameter, relating column height to the span length
b	=	nondimensional parameter, relating the total rise of the rafter to the length of span
c	=	nondimensional parameter, relating location of gable in saw-tooth frame to the span length
f, g	=	function values
K, K_1, K_2, K_i	=	nondimensional parameter, relating the fully plastic moment values of different spans
w	=	distributed vertical load per unit length at ultimate load
w_w	=	distributed vertical load per unit length at working load
A_1, A_2, A_3	=	nondimensional parameter, relating the horizontal force acting on a structure (or the hypothetical "overturning" moment of one part of a structure on the adjacent part) to the vertical loads
C	=	constant
$_nC_r$	=	number of possible combinations of hinges which result in failure of the structure

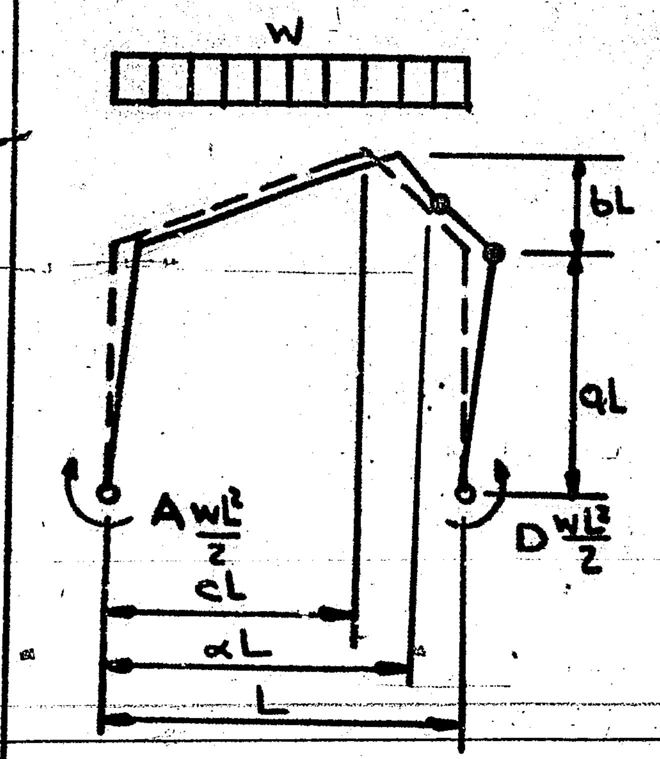
D_1, D_2, D_3	=	nondimensional parameter, relating the horizontal resisting force or hypothetical "overturning" moment acting on a structure to its vertical loading.
H	=	horizontal load
L, L_1, L_2, L_3	=	length measurement. Can be total span length or fractional part of it.
L_i	=	total length of member in frame
M_p	=	fully plastic moment value
P	=	concentrated horizontal load
V	=	shear force
W	=	weight per unit length of a structural member
W_{ext}	=	external work associated with a virtual displacement of an assumed mechanism
W_{int}	=	internal work associated with a virtual displacement of an assumed mechanism
$\alpha, \alpha_1, \alpha_2$	=	nondimensional parameters, defining the distance to the plastic hinge in the rafter of a structure
$\theta, \theta_1, \theta_2$	=	virtual rotation
	=	weight function

APPENDIX A

SUMMARY OF EXPRESSIONS FOR POSSIBLE FAILURE MODES

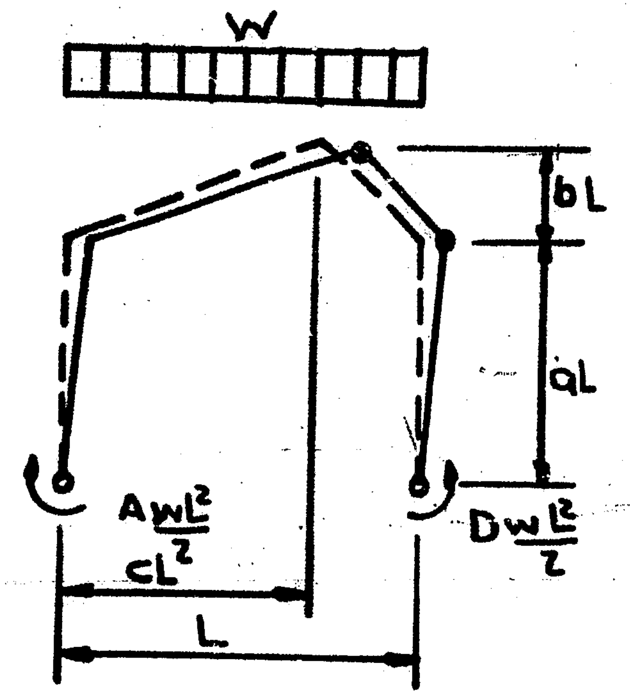


Above are shown all of the possible failure modes. Those denoted by S.S. were found to govern for $c = 0.7$ in the single span case and those denoted by M.S. were found to control in the multiple span case for $\frac{b}{a} = 1.0$ and $c = 0.7$.

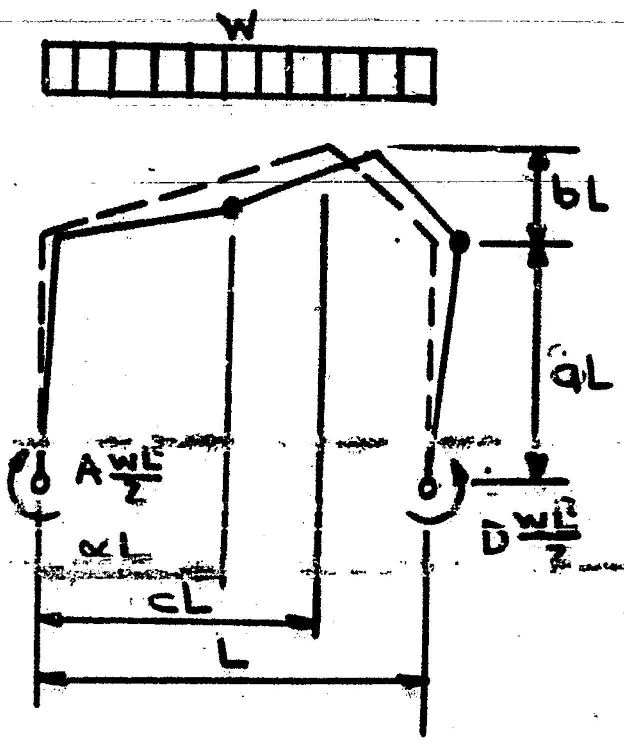


$$\frac{M_p}{wL^2} = \frac{\alpha}{2}\left[\frac{1-\alpha-D\left(1+\frac{b}{a}\frac{1}{c}\right)+A}{2+\frac{b}{a}\frac{\alpha}{c}}\right]$$

$$\alpha = \frac{-2+\sqrt{4-2\frac{b}{a}\frac{1}{(1-c)}\left[D\left(1+\frac{b}{a}\frac{1}{(1-c)}\right)-A-1\right]}}{\frac{b}{a}\left(\frac{1}{1-c}\right)}$$

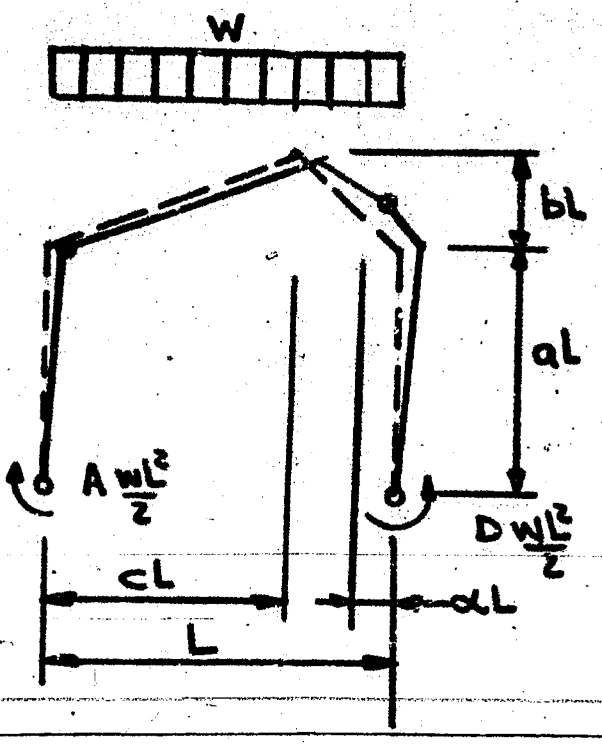


$$\frac{M_p}{wL^2} = \frac{1}{2}\left[\frac{(A+c)(1-c)-D\left(1+\frac{b}{a}-c\right)}{3+\frac{b}{a}-2c}\right]$$



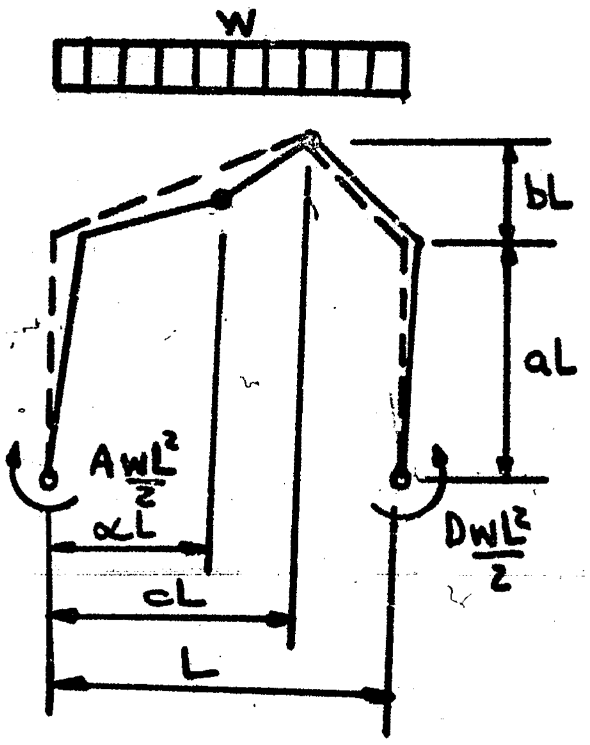
$$\frac{M_p}{wL^2} = \frac{1}{2}\left[\frac{(A+\alpha-D)(1-\alpha)-D\left(\frac{b}{a}\frac{\alpha}{c}\right)}{2+\frac{b}{a}\frac{\alpha}{c}}\right]$$

$$\alpha = \frac{1}{\frac{1}{2c}\frac{b}{a}}\left[\sqrt{1-\frac{1}{2c}\frac{b}{a}\left[A\left(1+\frac{1}{2c}\frac{b}{a}\right)-D\left(1-\frac{1}{2c}\frac{b}{a}\right)-1\right]}-1\right]$$



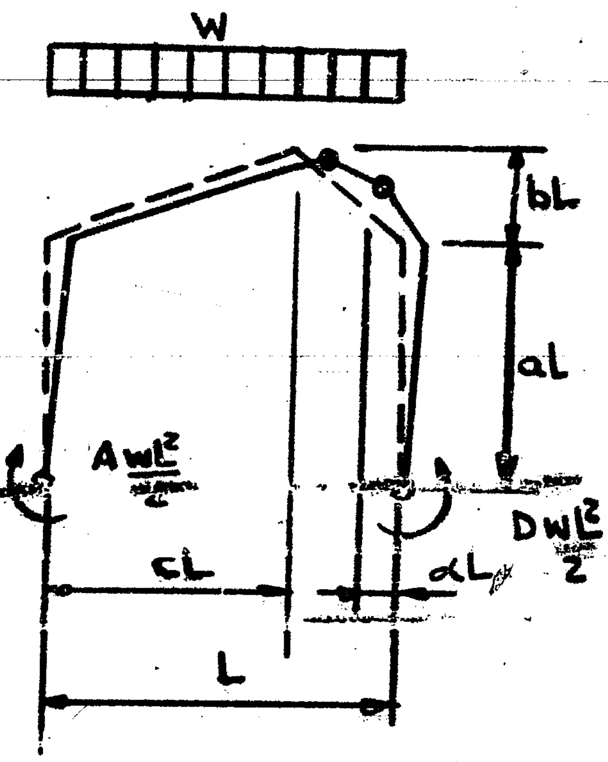
$$\frac{M_P}{wL^2}=\frac{1}{2}\left[\frac{(A-D-\alpha)(1-\alpha)+A\frac{b}{a}\frac{\alpha}{1-c}}{2+\frac{b}{a}\frac{\alpha}{(1-c)}}\right]$$

$$\alpha=\frac{\left[4\frac{a}{b}(1-c)+\frac{b}{a}\frac{1}{(1-c)}\right]+\sqrt{\left[4\frac{a}{b}(1-c)+\frac{b}{a}\frac{1}{(1-c)}\right]^2-4\left[(2D-2A-2)\left(\frac{b}{a}\frac{1}{(1-c)}\right)+A+D\right]}}{2}$$



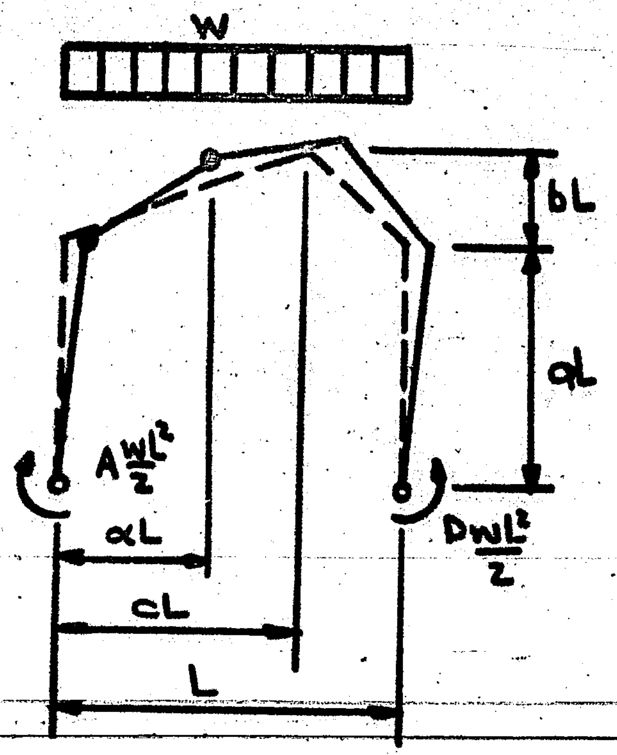
$$\frac{M_P}{wL^2}=\frac{1}{2}\left[\frac{A\left(\frac{b}{a}+c\right)+(c^2-c)-\alpha^2\left(\frac{b}{a}+1\right)+\alpha\left(\frac{b}{a}c+1-\frac{A}{c}\frac{b}{a}-A+D\right)-Dc}{2+\frac{b}{a}\left(1+\frac{\alpha}{c}\right)}\right]$$

$$\alpha=\frac{\sqrt{\left(\frac{b}{a}+2\right)^2-\frac{2b}{ca}\left[A-D+\frac{A}{c}\frac{b}{a}-1-\frac{c}{2}\frac{b}{a}\right]}-\left(2+\frac{b}{a}\right)}{\frac{b}{ac}}$$



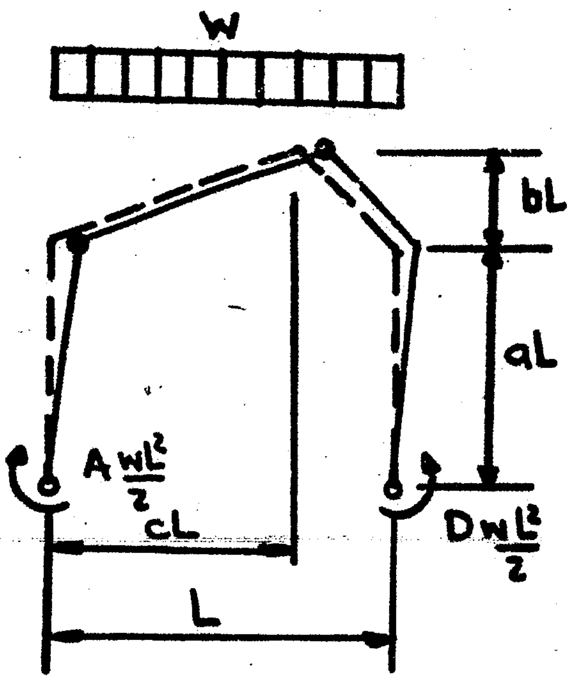
$$\frac{M_P}{WL^2}=\frac{1}{2}\left[\frac{(A+c-c\alpha)\left(1+\frac{b}{a}\frac{\alpha}{1-c}\right)(1-c)-\alpha\left(1+\frac{b}{a}\right)}{2+\frac{b}{a}\left(1+\frac{\alpha}{1-c}\right)}\right.$$

$$\left.\frac{(2\alpha^2+c\alpha-\alpha+D)\left[\left(1+\frac{b}{a}\right)(1-\alpha)-c\left(1+\frac{b}{a}\frac{\alpha}{1-c}\right)\right]}{}\right]$$

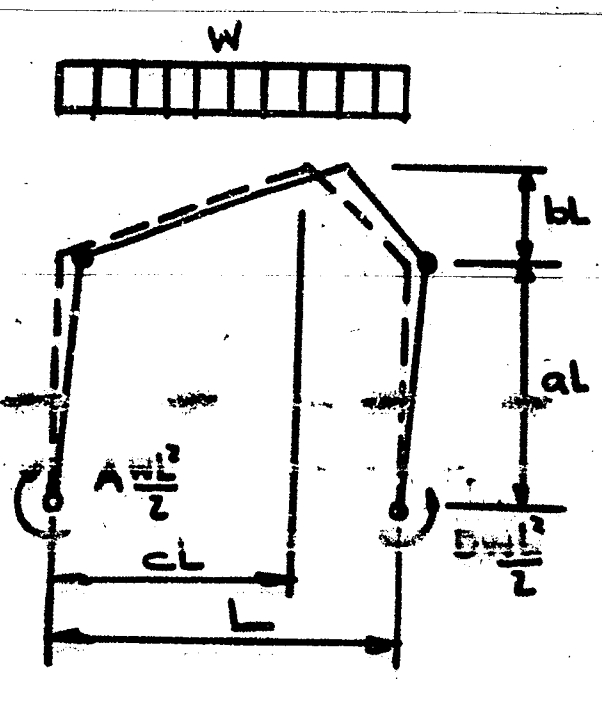


$$\frac{M_p}{wL^2} = \frac{\alpha}{2}\left[\frac{A\left(1+\frac{b}{a}\frac{1}{c}\right) - D - (1-\alpha)}{2+\frac{b}{a}\frac{\alpha}{c}}\right]$$

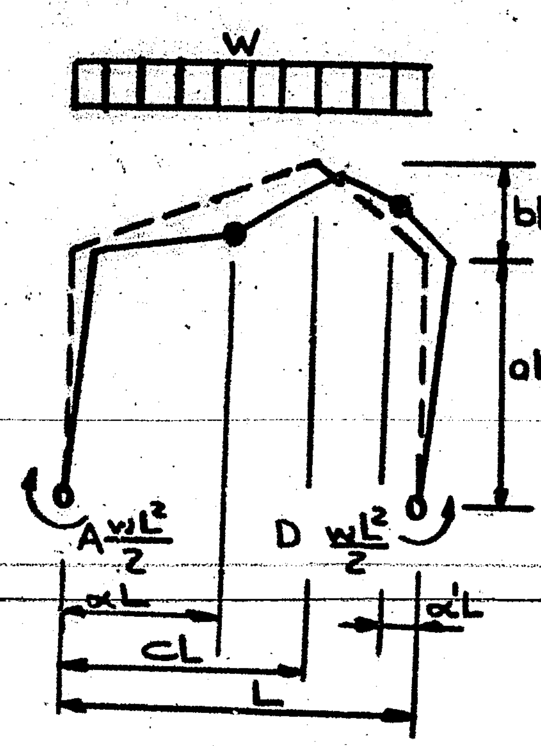
$$\alpha = \frac{-2+\sqrt{4-2\frac{b}{a}\frac{1}{c}\left[A\left(1+\frac{b}{a}\frac{1}{c}\right)-D-1\right]}}{\frac{b}{a}\frac{1}{c}}$$



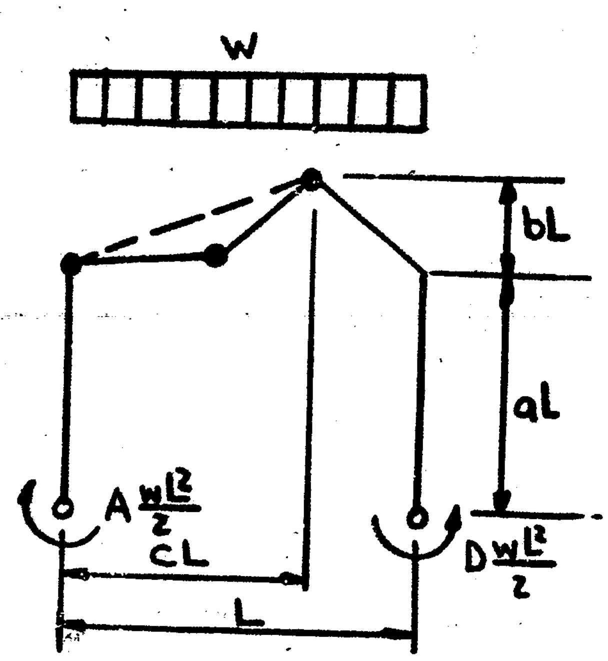
$$\frac{M_p}{wL^2} = \frac{c}{2}\left[\frac{A\left(1+\frac{b}{a}\frac{1}{c}\right) - D - (1-c)}{2+\frac{b}{a}}\right]$$



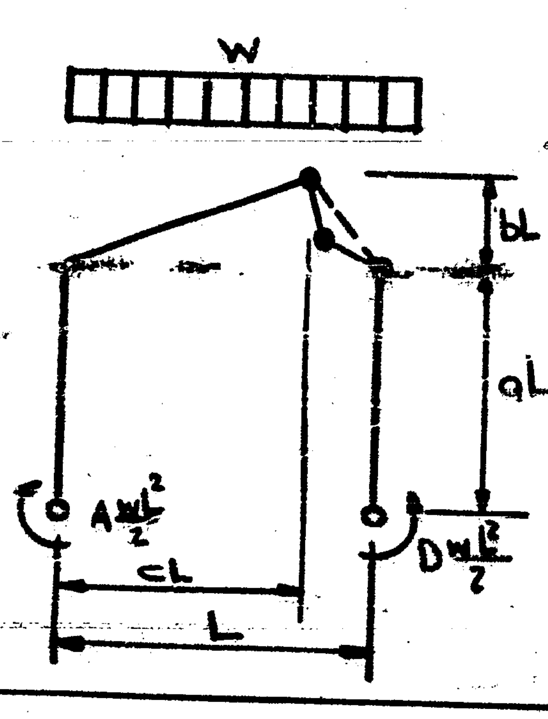
$$\frac{M_p}{wL^2} = \frac{1}{4}(A-D)$$



$$\frac{M_p}{wL^2} = \frac{1}{4}\left[\frac{\frac{b}{a}\alpha\alpha'\left(\frac{\alpha}{1-c}+\frac{\alpha'}{c}\right)+\frac{b}{a}\frac{\alpha\alpha'}{c(1-c)}\left(1-2\alpha-2\alpha'+2\alpha\alpha'\right)+\left(\alpha-\alpha^2-4\alpha\alpha'+2\alpha^2\alpha'+\alpha'-\alpha'^2-2\alpha\alpha'^2\right)-D(1-\alpha-\alpha')-\frac{b}{a}\frac{\alpha D}{c}+\frac{D\alpha\alpha' b}{c(1-c)}+A\left[(1-\alpha-\alpha')+\frac{b}{a}\frac{\alpha'}{(1-c)}-\frac{b}{a}\frac{\alpha\alpha'}{c(1-c)}\right]}{2+\frac{b}{a}\frac{\alpha}{c}+\frac{b}{a}\frac{\alpha'}{(1-c)}}\right]$$



$$\frac{M_p}{wL^2} = \frac{c^2}{16}$$



$$\frac{M_p}{wL^2} = \frac{(1-c)^2}{16}$$

APPENDIX B

SAMPLE DESIGN CHARTS

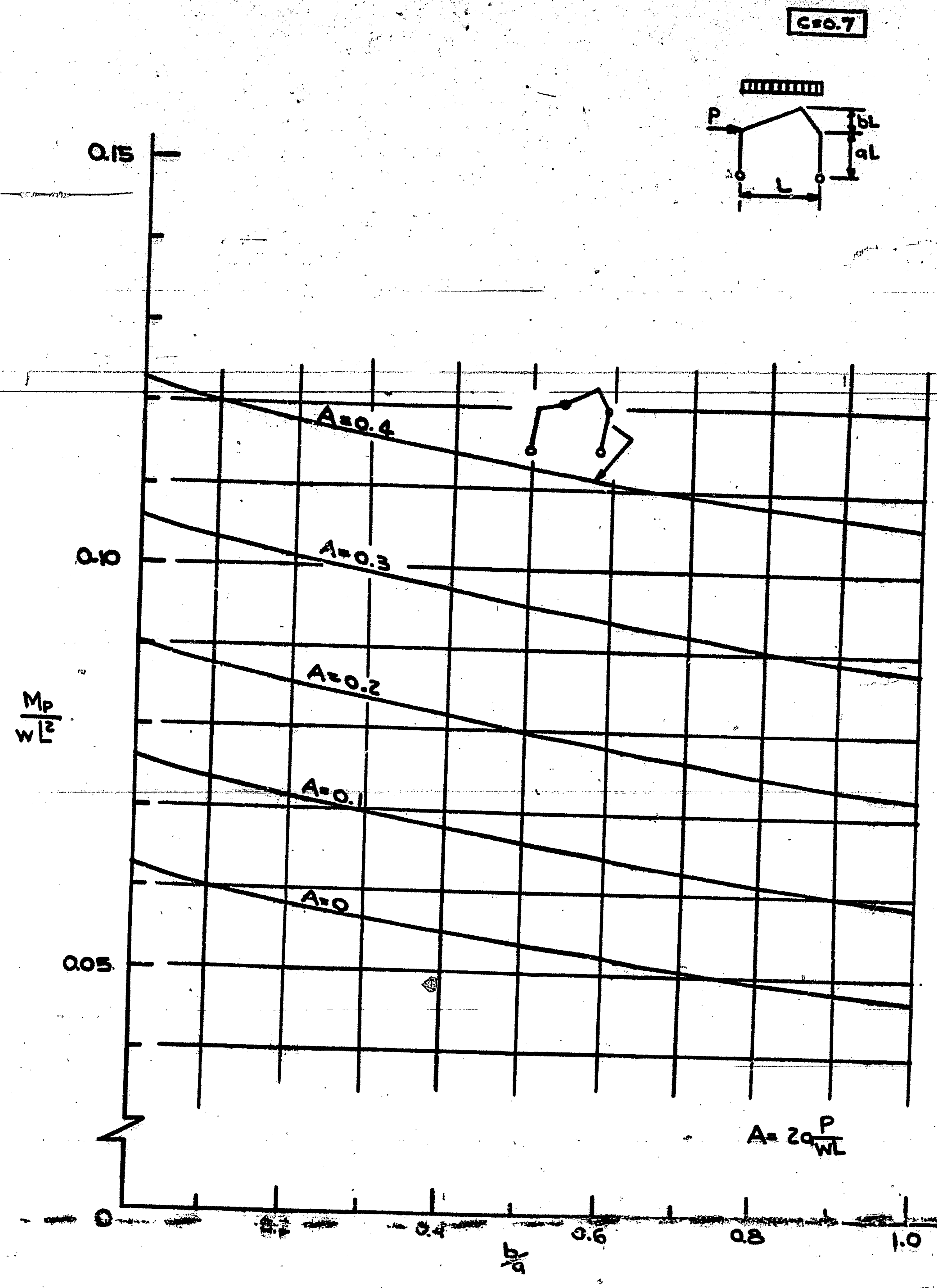


FIGURE 26 - DESIGN CURVES FOR PINNED-BASE, SAW-TOOTH FRAMES DETERMINATION OF MEMBER SIZE c = 0.7

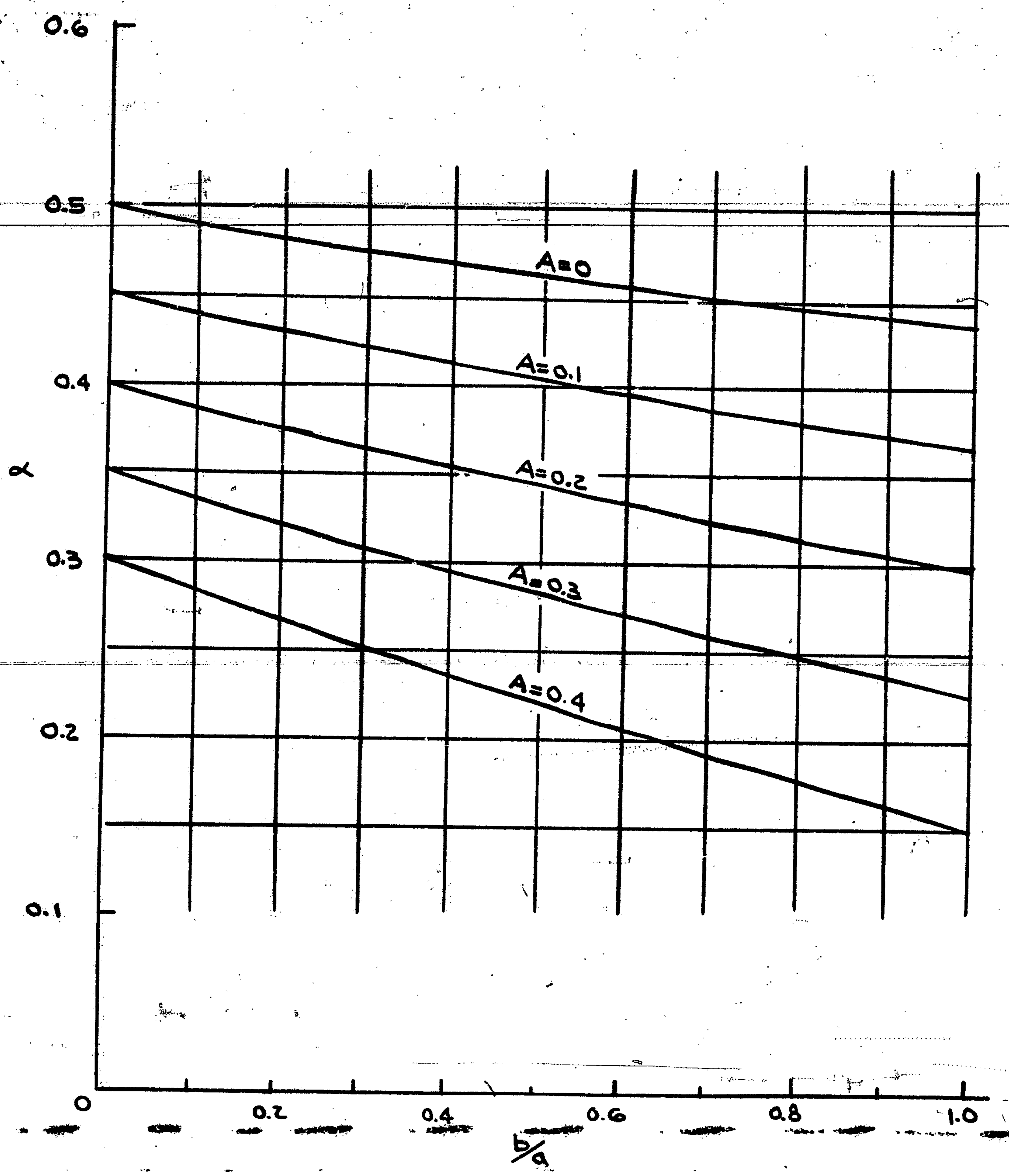


FIGURE 27 - DESIGN CURVES FOR PINNED-BASE, SAW-TOOTH FRAME LOCATION OF PLASTIC HINGE c = 0.7

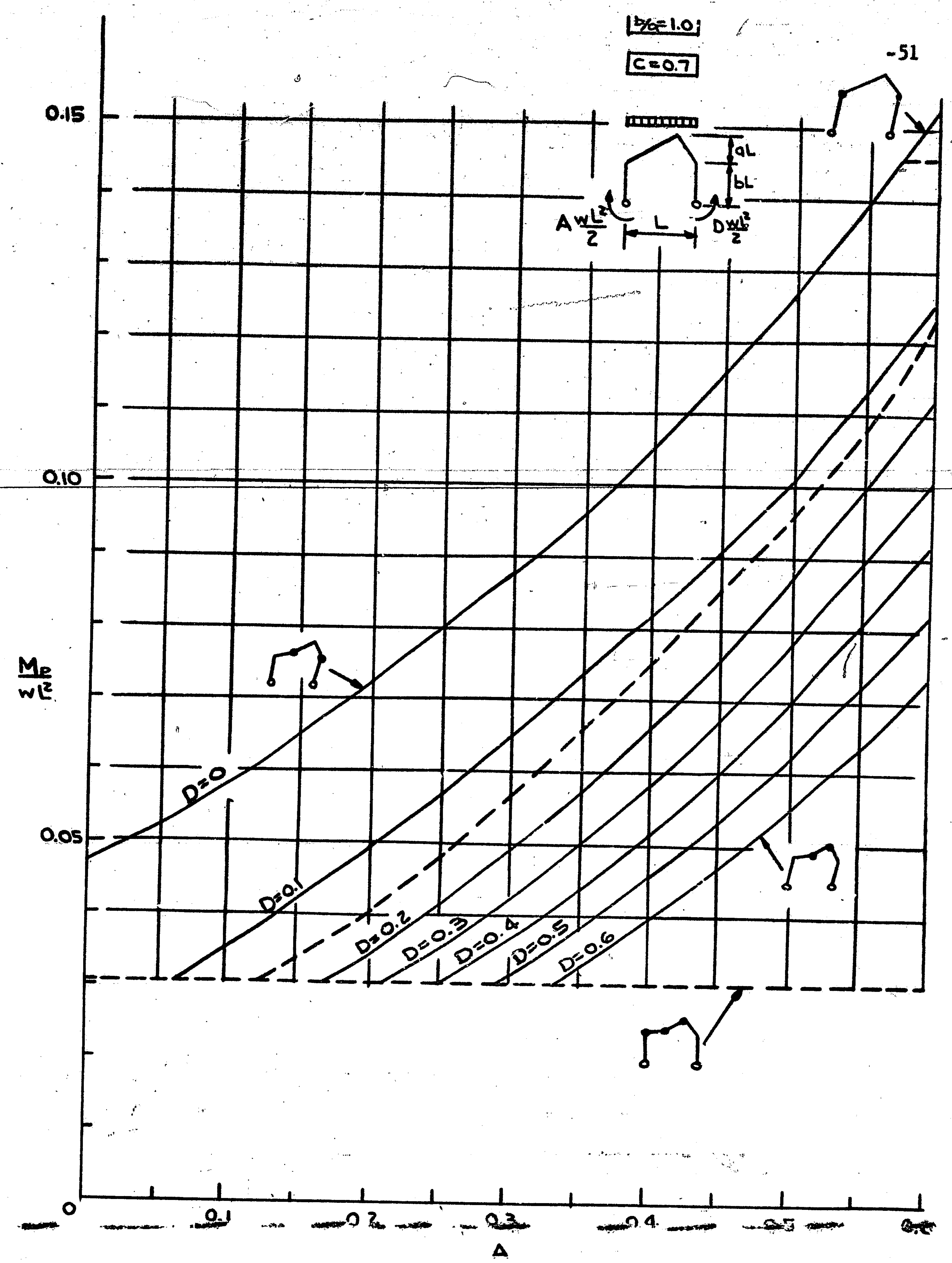


FIGURE 28 - DESIGN CURVES FOR PINNED-BASE, SAW-TOOTH FRAMES
DETERMINATION OF MEMBER SIZE c = 0.7 b/a = 1.0

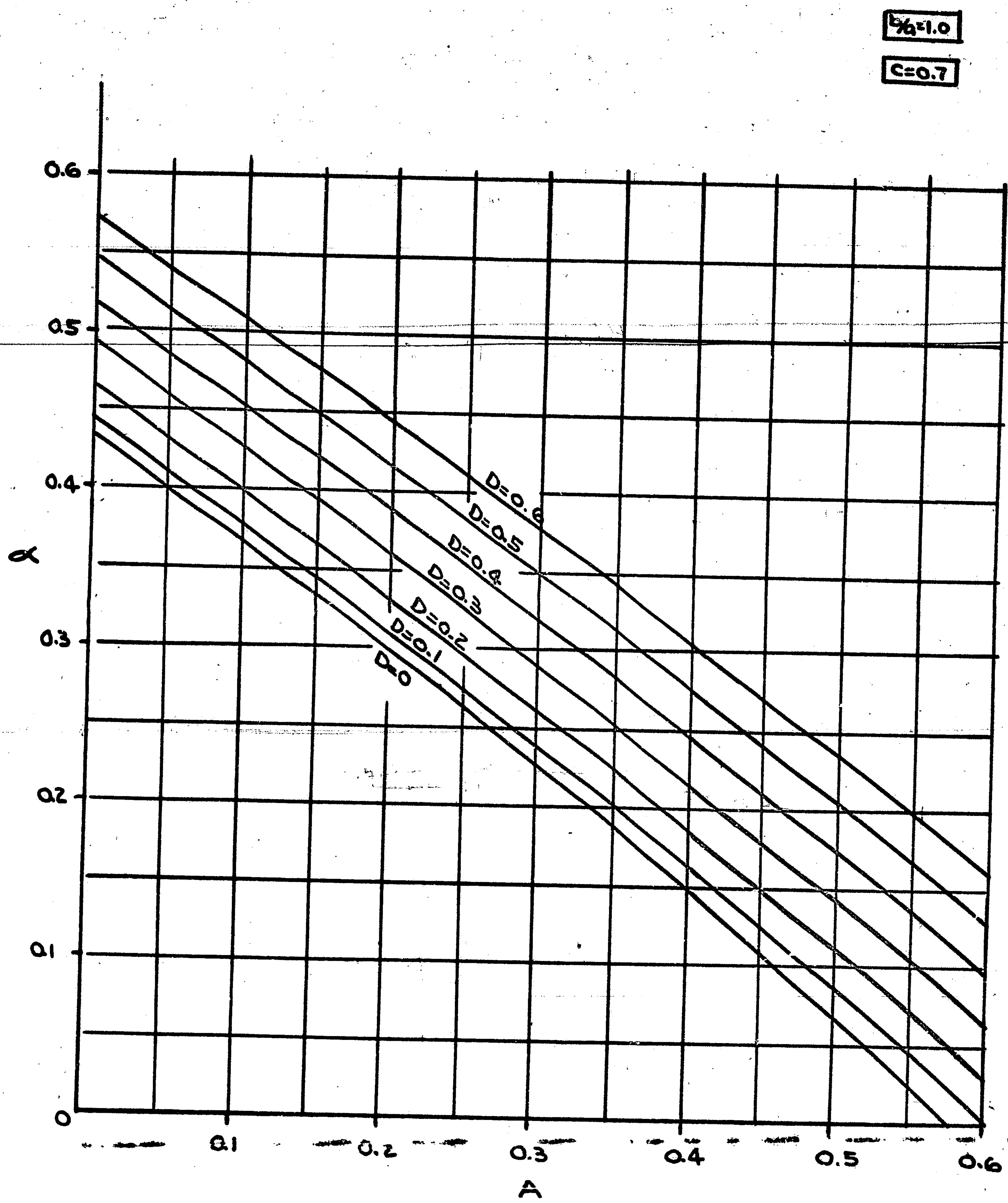


FIGURE 29 - DESIGN CURVES FOR PINNED-BASE, SAW-TOOTH FRAME LOCATION OF PLASTIC HINGE $c = 0.7$ $\frac{b}{a} = 1.0$

V I T A

The author was born the son of Santy and Mary Recchio on the twenty-second day of January, 1935, at Canton, Ohio. He graduated from Timken Vocational High School, Canton, Ohio, in June, 1952. After a short period of employment he entered the Georgia Institute of Technology of Atlanta, Georgia where he was enrolled as a student in the co-operative plan and received the Bachelor of Civil Engineering in June, 1958. He entered the Graduate School of Lehigh University in September, 1958, and accepted a position as a part-time research assistant at the Fritz Engineering Laboratory while preparing for the Degree of Master of Science.